REPORT ON NON-COMMUTATIVE PROBABILITIES AND VON NEUMANN ALGEBRAS

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We sought to understand non-commutative probabilities from a model-theoretic point of view via the study of von Neumann algebras equipped with a finite trace.

We formalised the unit balls of such structures as continuous structures and showed they formed an elementary class. The theory T_0 axiomatising this class can be taken to be universal if we add a unary function symbol $\dot{2}$ such that if $\|a\| \leq \frac{1}{2}$ then $\dot{2}a = 2a$.

The following sets were shown to be definable:

(1) The set of all self-adjoint elements.
(2) The set of all projections.
(3) The set of all projections below a given projection p (uniformly in p).
(4) The set of all projections such that $\tau(p) = \frac{1}{2}$.

Using these facts we can explicitly write an axiom NA saying that the von Neumann algebra is atomless, and modulo that, an axiom NC saying it is centreless (i.e., a II_1 factor). Furthermore, one also shows the sets above are in fact inf-definable, from which it follows that the new axioms NA and NC are $\forall\exists$. Thus the theory $T_1 := T_0 + NA + NC$, whose models are precisely the II_1 factors, is an $\forall\exists$-theory. As T_0 and T_1 are companions (every model of one embeds in one of the other), we conclude that every existentially closed (e.c.) von Neumann algebra (with a finite trace) is a II_1 factor.

We can further show that the class of von Neumann algebras which embed in some ultrapower of R is elementary, axiomatised by a universal theory T_0^e.

0.1. **Question.** *Find explicit axioms for T_0^e.*

This is a universal theory, although we still have to find an explicit statement of the embeddability axiom. We know that T_0^e and $T_1^e := T_1 \cup T_0^e$ are companions, so every e.c. embeddable von Neumann algebra is an embeddable II_1 factor.

0.2. **Question.** *Does T_1^e admit quantifier elimination?*

A reasonable plan of approach would be to show that $R^{\mathscr{U}}$ admits quantifier-free back-and-forth. There are indications this may be true, but more work is required.

Since $R^{\mathscr{U}}$ is $\aleph_1$-saturated, this means that $\mathrm{Th}(R^{\mathscr{U}}) = \mathrm{Th}(R)$ eliminates quantifiers. This theory clearly contains T_1^e. Are they equal?

0.3. **Question.** *Do e.c. models of T_1^e admit a "nice" notion of independence?*

The theory T_1^e has the independence property, and is therefore unstable. There is still a vague hope it might be simple.

There exists many possible notions of independence in von Neumann algebras, the weakest of which is the following:

$$A \mathop{\smile\!\!\!\!\!\!|}_{B} C \iff \big(\forall a \in \langle AC\rangle\big)\big(\forall b \in \langle BC\rangle\big)\big(\ {}_{\langle C\rangle}(ab) = \ {}_{\langle C\rangle}(a) \ {}_{\langle C\rangle}(b)\big)$$
$$\iff \big(\forall a \in \langle AC\rangle\big)\big(\ {}_{\langle C\rangle}(a) = \ {}_{\langle BC\rangle}(a)\big).$$

Here ${}_{\mathscr{B}}$ denotes conditional expectation with respect to $\mathscr{B}$, i.e., orthogonal projection in the sense of the inner product $\langle x, y\rangle = \tau(x^*y)$.

We verified this notion of independence satisfies all axioms of a stable/simple notion of independence expect for stationarity/independence theorem. Indeed, it seems that with stronger notions of independence we may run into trouble with the local character. While stationarity is known to fail (by direct constructions, as well as from the fact that the theory is unstable), there may still be hope the independence theorem holds. If that is the case then the theory is simple with $\mathop{\smile\!\!\!\!\!\!|}$ coinciding with non-dividing.