Cohomological methods in abelian varieties

The American Institute of Mathematics

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Version: Tue Feb 28 18:18:16 2012

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Chapter A: Participant Contributions

A.1 Berger, Lisa

In recent work I describe a construction of surfaces $\mathcal{S} \to \mathbb{P}^1_k$, dominated by products of curves, with the property that they remain so under base extension. This result is combined with work of Doug Ulmer to produce a parameterized family of elliptic curves which obtain unbounded ranks over the fields $\mathbb{F}_p(t^{1/d})$, as $d = p^n + 1$ grows arbitrarily large. More recently, Ulmer works to explicitly describe the geometry in our construction, and he proves a formula for the ranks of my curves over $k(t)$, k algebraically closed of arbitrary characteristic. My own construction was motivated by work of Shioda and Katsura, who show that a Fermat variety is dominated by a product of lower dimensional Fermat varieties. My primary goal is to substantially improve my depth of understanding of the geometry in Ulmer's work and also that of Shoida on Fermat varieties. I believe that other questions could be asked, and answered, about my construction. Indeed, I think that any question that has been pursued about algebraic cycles on Fermat varieties and their quotients should be studied for other product varieties. This is a vague idea. I appreciate the opportunity to learn from the more experienced researchers attending this workshop. I want to learn to formulate–and perhaps even answer–good questions.

A.2 Cesnavicius, Kestutis

I am interested in cohomological obstructions to existence of rational points on algebraic varieties defined over number fields, such as the Brauer-Manin obstruction. An example of a particular open question that interests me is the conjecture of Colliot-Thelene that the Brauer-Manin obstruction is the only obstruction to the existence of a zero cycle of degree one. More precisely, suppose X is a smooth, projective, geometrically integral variety defined over a number field k and let $\{z_\nu\}_{\nu\in\Omega}$ be a collection of local zero cycles $z_\nu \in Z_0(X_\nu)$ of degree one. (Here Ω is the set of places of k, for $\nu \in \Omega$ we let X_ν be the base change of X to the completion k_ν of k at ν, $Z_0(X_\nu)$ is the free abelian group on the closed points of X_ν, and the degree of $z_\nu = \sum n_i z_i^\nu \in Z_0(X_\nu)$ is $\sum n_i[k(z_i^\nu) : k_\nu]$ where $k(z_i^\nu)$ is the residue field of the closed point z_i^ν.) The conjecture then predicts that if $\{z_\nu\}$ pairs to zero with every element of the Brauer group of X, i.e., if $\sum_\nu \mathrm{inv}_\nu A(z_\nu) = 0$ for each $A \in \mathrm{Br}(X)$, then there exists a degree one zero cycle on X. As first explained by Skorobogatov, the Brauer-Manin obstruction is not the only obstruction to the existence of a rational point (which in particular is a degree one zero cycle), however, the conjecture claims that this can be remedied if rational points are replaced by degree one zero cycles. The conjecture fits in a more general web of conjectures made by Colliot-Thelene which are formulated in terms of the cycle class map in étale cohomology exploiting the Poitou-Tate-Saito exact sequence. It seems natural to hope that a good understanding of cohomological methods would allow one to appreciate these conjectures better, as well as to understand previous approaches to their special cases which could perhaps be expanded to a wider class of varieties.

A.3 Esnault, Helene

I'm not a specialist of abelian varieties. Here are 2 points:

I - Existence of ℓ'-companions (Deligne's conjecture in Weil II): Drinfeld (using Deligne's number field result) proved it for X smooth over $\mathbb{F}_q$, while it is conjectured for X normal. One can compute on X normal that if $A- > X_{reg}$ is an abelian scheme over its smooth locus, and the associated ℓ-adic lisse sheaf over X_{reg} extends as a lisse sheaf over X, then so does the ℓ'-adic sheaf (discussions with Pierre Deligne and Moritz Kerz)

II - If u is an automorphism of an abelian surface over a finite field, one can use the Tate conjectures to show that the absolute value of the eigenvalues of u^* on ℓ-adic cohomology are bounded above by the ones on the orbit of the class of some polarization in the Néron-Severi (joint with V. Srinivas). It would be very nice if someone could generalize this to higher dimensional abelian varieties.

A.4 Gorchinskiy, Sergey

Let M be an integral Chow motive such that its Betti realization is $G[-1]$, where G is a free abelian group of finite rank. Does there exist an operation $M \mapsto \lambda^2(M)$ on Chow motives such that the Betti realization of $\lambda^2(M)$ is equal to $\wedge^2(G)[-2]$? Does there exist an analogue of this for other wedge powers? Does there exist an analogue of this for the case, when the Betti realization of M is $G[d]$ for some integer d (and the wedge power is replaced by an appropriate symmetric power or something else)? It seems possible that in order to define such an operation one needs to pass from the category of Chow motives to the Morel-Voevodsky motivic stable homotopy category.

Given an abelian variety A, does there exist an integral Chow motive which should be called $M^1(A)$? Note that it exists when A is a Jacobian of a curve. If $M^1(A)$ exists, what is a relation between the above operations applied to $M^1(A)$ and the integral motive of A?

Having realized this, does the integral Kimura finite-dimensionality of the motive $M^1(A)$ (which is still to be defined) lead to conjectures about finiteness of the Brauer group for abelian varieties over artihmetic type field?

A.5 Harari, David

I am especially interested in questions related to the Brauer-Manin obstruction to the existence or the density of rational points on algebraic varieties. Many problems remain open for fibrations in elliptic curves over the projective line (like certain K3 surfaces), and I hope that this workshop will be a good occasion to discuss these topics with well-known experts.

Also in the last few years I have worked on topics related to arithmetic duality theorems for abelian varieties (and more generally, 1-motives); this is linked to the question of divisibility of elements of the Tate-Shafarevich group in the Weil-Chtelet group; recent progress has been made recently by Ciperiani and Stix on this question, which I expect to discuss at the conference.

A.6 Kahn, Bruno

It is known that the Tate conjecture over finite fields implies the generalised Tate conjecture, and that the Hodge conjecture for CM abelian varieties implies the generalised Hodge conjecture for the same class of varieties. Can we make the arguments in the corresponding proofs effective? A good testing ground is products of elliptic curves, for which both the Hodge and the Tate conjecture are known. I studied this in the case of the Tate conjecture in arXiv:1101.1730. Since then, things have clarified a little more and the Hodge parallel works very well. Here is a summary:

A finite collection (E_i) of elliptic curves over $k = C$ or F_q is "in general position" if the endomorphism fields of the CM ones (over C) or the ordinary ones (over F_q) are linearly disjoint. (There may be many isogenous E_i in a family in general position: the condition is on the set of distinct endomorphism fields.) If (E_i) is in general position, the generalised Hodge resp. Tate conjecture holds for their product.

As a consequence, the generalised Hodge resp. Tate conjecture holds for N^1H^3 of the product for any family, in general position or not. (Note that 3 distinct quadratic imaginary fields must be linearly disjoint.)

The first interesting case is that of 4 CM resp. ordinary elliptic curves in special position. Actually, let's start with three, E_1, E_2, E_3, in general position. Let K_i be the endomorphism field of E_i and $K = K_1K_2K_3$. Then the triquadratic extension K/Q contains exactly one 4th quadratic imaginary subfield K_0; to K_0 corresponds an isogeny class of CM resp. ordinary elliptic curves. Let's choose a representative E_0.

Let $B = E_0\times E_1\times E_2\times E_3$. It turns out that the generalised Hodge resp. Tate conjecture for $N^1H^4(B)$ is controlled by a Hodge resp. Tate cycle of codimension 3 on $B \times A^2$, where A is a simple CM resp. ordinary abelian variety of dimension 4, with endomorphism field K. Replacing B by another abelian variety, one may convert this Hodge cycle into a Weil cycle (an old note of Yves André gives the recipe).

The Hodge resp. Tate conjecture holds for A and its powers in the strong sense that Hodge resp. Tate classes are generated in codimension 1, just as for products of elliptic curves. This is far from being the case for $B \times A^2$. In fact, the situation in the Hodge case is as follows: there is a diagram of Mumford-Tate groups

$$MT(B) \longleftarrow MT(A \times B) \longrightarrow MT(A)$$

where both maps are isogenies of tori, with kernel of order 2.

It looks like an interesting challenge to prove that the corresponding Hodge or Tate cycle is algebraic [or to disprove it]...

A.7 Moonen, Ben

For Weil-type abelian varieties of dimension 4, there are two cases where non-trivial algebraic cycles of codimension 2 have been found. Schoen did this for the CM field $\mathbb{Q}(\zeta_3)$; he describes the abelian variety as a Prym. This seems hard to generalize to higher dimensions. Van Geemen gave a construction for the field $\mathbb{Q}(i)$; he obtained the desired cycle in a geometric way, using relations between theta functions. Of course, these are very special examples, in that we have an automorphism α of the polarized abelian variety.

First question: Can we do produce more examples following van Geemen's method, maybe in higher dimensions? How does one make explicit the action of the given automorphism α (in van Geemen's case of order 4) on the linear system $|2\Theta|$? Of course "Weil representation" is the relevant notion here, but I'm asking how to make this explicit once we know the action of α on the 2-torsion of X.

Second question: In the known examples, can we write down a vector bundle whose c_2 is an exceptional class?

Related to this: If we describe our abelian varieties as complex tori and we look at vector bundles, is there any method to choose canonical automorphy factors, or to make a particularly good choice? Further, if we have a description of a vector bundle in terms of automorphy factors, how do we calculate from this the Chern classes? There is work by Brylinsky and McLaughlin where they do this in Cech cohomology; it seems worthwile to adapt this to the context considered here.

A.8 Pannekoek, Rene

I am a PhD student who is interested in rational points on K3 surfaces. I am participating in this workshop primarily because of my interest in Brauer groups of K3 surfaces, methods to calculate these groups, and any topics that might be related to this.

A.9 Polishchuk, Alexander

I am interested in Chow groups of abelian varieties in general and more recently in questions related to integral structures. For example, I'd like to understand whether the Chow ring $CH(J_C)$, where J_C is a Jacobian of a curve over an algebraically closed field, or perhaps its localization at 2, admits divided powers (this question was asked by Bruno Kahn). In a joint work with Ben Moonen, using the Fourier-Mukai transform, we studied the case when C is hyperelliptic but the case of an arbitrary Jacobian is still open. The related question is whether there exists an integral version of the Fourier-Mukai transform for J_C.

A.10 Poonen, Bjorn

I am interested in the arithmetic aspects of abelian varieties (Selmer groups, etc.), and in cohomological obstructions to rational points on varieties in general.

A.11 Rubin, Karl

I am especially interested in Selmer groups of elliptic curves and abelian varieties, and their variation in families.

A.12 Shlapentokh, Alexandra

I am not a specialist in abelian varieties and I am hoping there will be introductory lectures on topics of the workshop. In general I am interested in using abelian varieties for the purposes of Diophantine definability. More specifically, I am interested in behavior of Mordell-Weil group of elliptic curves over infinite algebraic extensions of product formula fields. In particular, I am interested in elliptic curves with finitely generated groups over infinite algebraic extensions of $\mathbf{Q}$.

A.13 Schoen, Chad

I am generally interested in the topics of the workshop, but I do not have specific problems to suggest.

A.14 Silverberg, Alice

For the workshop, I'm primarily interested in arithmetic applications, including the behavior of the Mordell-Weil rank for abelian varieties over number fields and function fields, and the arithmetic of rational points on elliptic surfaces over number fields. (I am also interested in learning more about the other topics. Since I'm much less familiar with the other topics, I hope that there will be some introductory talks that are very accessible to people who are interested but are not experts.)

A.15 Skorobogatov, Alexei

This text arose in the context of an ongoing discussion between Bruno Kahn and the author.

Let n be a positive integer. Let A be an abelian variety over a field k whose characteristic is coprime to n. Let $\overline{k}$ be a separable closure of k, $\Gamma = \mathrm{Gal}(\overline{k}/k)$, and $\overline{A} = A \times_k \overline{k}$. We have the spectral sequence

$$\mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(\overline{A}, \mu_n)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(A, \mu_n). \tag{1}$$

Consider the following question:

Does the spectral sequence (1) *degenerate?*

It is not very hard to see that the answer is positive if A is a product of elliptic curves, and, more generally, a product of principal homogeneous spaces of elliptic curves whose classes in the corresponding Weil–Châtelet groups are all divisible by n.

Let us consider the canonical map

$$d^{0,2} : E_2^{0,2} = \mathrm{H}^2_{\text{ét}}(\overline{A}, \mu_n)^\Gamma \longrightarrow E_2^{2,1} = \mathrm{H}^2(k, \mathrm{H}^1_{\text{ét}}(\overline{A}, \mu_n)).$$

Assume that $n = \ell^m$, where ℓ is a prime not equal to the characteristic of k, and $m \geq 1$. By [**?**, Prop. 2.2] we see that for odd ℓ this map is zero. Now take $\ell = 2$. Let ι_n be the cycle class map

$$\iota_n : \mathrm{NS}(\overline{A})/n \hookrightarrow \mathrm{H}^2_{\text{ét}}(\overline{A}, \mu_n).$$

We write A^t for the dual abelian variety of A. Then we have a canonical isomorphism of Γ-modules $A^t_n = \mathrm{H}^1_{\text{ét}}(\overline{A}, \mu_n)$.

Lemma 0.1. *The composition* $d^{0,2}\iota_2 : (\mathrm{NS}(\overline{A})/2)^\Gamma \to \mathrm{H}^2(k, A_2)$ *is the connecting homomorphism of the natural extension*

$$0 \to A^t_2 \longrightarrow \mathrm{Pic}(\overline{A}) \xrightarrow{[2]} \mathrm{Pic}(\overline{A}) \longrightarrow \mathrm{NS}(\overline{A})/2 \to 0, \tag{2}$$

where the middle map is multiplication by 2.

Proof This is proved in the same way as [**?**, Lemme 3.1]. QED

We have the following commutative diagram

$$\begin{array}{ccccccccccc} 0 & \to & A^t_2 & \to & \mathrm{Pic}(\overline{A}) & \xrightarrow{[2]} & \mathrm{Pic}(\overline{A}) & \to & \mathrm{NS}(\overline{A})/2 & \to & 0 \\ & & || & & \uparrow & & || & & \uparrow & & \\ 0 & \to & A^t_2 & \to & A^t & \xrightarrow{[2]} & \mathrm{Pic}(\overline{A}) & \to & \mathrm{NS}(\overline{A}) & \to & 0 \end{array} \tag{3}$$

where the left hand vertical map is the natural injection, and the right hand one is the natural surjection. Let

$$\partial : \mathrm{NS}(\overline{A})^\Gamma \to \mathrm{H}^2(k, A^t_2)$$

be the connecting homomorphism attached to the bottom 2-extension of (3). To prove that $d^{0,2}$ is not the zero map, by Lemma 0.1 it is enough to find a Galois-invariant class $\lambda \in \mathrm{NS}(\overline{A})^\Gamma$ such that $\partial(\lambda) \neq 0$. The bottom 2-extension of (3) is obtained by splicing the following two obvious 1-extensions:

$$0 \to A^t_2 \to A^t \xrightarrow{[2]} A^t \to 0, \tag{4}$$

and

$$0 \to A^t \to \mathrm{Pic}(\overline{A}) \to \mathrm{NS}(\overline{A}) \to 0. \tag{5}$$

Let $\partial_1 : \mathrm{NS}(\overline{A})^\Gamma \to \mathrm{H}^1(k, A^t)$ and $\partial_2 : \mathrm{H}^1(k, A^t) \to \mathrm{H}^2(k, A^t_2)$ be the corresponding connecting maps. Since $\partial = \partial_2\partial_1$, to achieve our goal it is enough to find a field k, an abelian variety

A and an element $\lambda \in \mathrm{NS}(\overline{A})^\Gamma$ such that $c_\lambda := \partial_1(\lambda)$ is not divisible by 2 in $\mathrm{H}^1(k, A^t)$. (It is easy to see that $2c_\lambda = 0$ for any $\lambda \in \mathrm{NS}(\overline{A})^\Gamma$, see [**?**, p. 1119].)

This cannot be done if k is local field, because in this case $c_\lambda = 0$ for any $\lambda \in \mathrm{NS}(\overline{A})^\Gamma$ and any A, see [**?**, Lemma 1].

This cannot be done if k is a number field, because in this case c_λ belongs to the Tate–Shafarevich group of A by [**?**, Cor. 2], and every element of the Tate–Shafarevich group is divisible by any prime number in $\mathrm{H}^1(k, A)$, see [Milne].

Now let $A = J$ be the Jacobian of a smooth, projective and geometrically integral curve X of genus $g \geq 1$, and let $\lambda \in \mathrm{NS}(\overline{J})^\Gamma$ be the class of the canonical principal polarisation of J. By [**?**, Cor. 4] we have

$$c_\lambda = [\mathrm{Pic}^{g-1}_{X/k}] \in \mathrm{H}^1(k, J).$$

We now let $g = 2$.

Problem: Find a field k and a curve X of genus 2 over k such that the class $[\mathrm{Pic}^1_{X/k}]$ is not divisible by 2 in $\mathrm{H}^1(k, J)$.

Since the class of $Y = \mathrm{Pic}^1_{X/k}$ is annihilated by 2, we see that Y is a 2-covering of J. In particular, Y is the twist of J by a 1-cocycle $\xi \in \mathrm{H}^1(k, J_2)$.

If there exists a k-torsor Z under J such that $[Y] = 2[Z]$ in $\mathrm{H}^1(k, J)$, then Z is a 4-covering of J, i.e. Z is the twist of J by a 1-cocycle whose class is in $\mathrm{H}^1(k, J_4)$. To show that such a Z does not exist we must show that $\xi \in \mathrm{H}^1(k, J_2)$ is not in the subgroup generated by the images of $\mathrm{H}^1(k, J_4)$ and $J(k)/2$.

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A.16 Ulmer, Douglas

I'd like to learn more about the relevant Fourier-Mukai transforms and their applications to integral motives.

In recent joint work with Rachel Pries, we have encountered an interesting example that looks somhow like mirror symmetry in an arithmetic setting. I'd be interested in any comments on this example.

The example is an elliptic curve E over $\mathbb{F}_q(t)$ with a collection of points indexed by elements $a \in \mathbb{F}_q$. These points generate a subgroup of Mordell-Weil of rank $q-1$. Based on related examples, I would expect that the height pairing, up to scaling, would be

$$< P_a, P_b >= \begin{cases} q-1 & \text{if } a = b \\ -1 & \text{if } a \neq b. \end{cases}$$

In fact, what happens is that there is a correction factor which is controled by the number of points in fibers of a *different* elliptic fibration. More precisely, there is another elliptic curve E' over $\mathbb{F}_q(t)$ such that $< P_a, P_b >$ is as above plus (a scaling of) A_{a-b} where the number of points on the fiber of E' over $a-b$ is $q+1-A_{a-b}$.

In brief, intersection numbers on one surface are controlled by numbers of points in fibers of another surface.

For the moment, the proof is purely computational and doesn't reveal why the second surface comes in, nor does it seem to give an a priori prediction of what the second surface is.

A.17 Voisin, Claire

Question 1. Let A be an abelian variety of dimension g defined over a number field K. For any point $a \in A(K)$, the 0-cycle $a - 0$ is of degree 0, and thus its etale cycle map gives an element

$$AJ(a-0) \in H^1(Gal(\overline{K}/K), H^{2g-1}_{et}(A_{\overline{K}}, Q_l(g))).$$

The first question I would like to understand better is the following:

What is the image of AJ?

This may be well-known. I simply do not know the answer.

I have the following related question in mind:

Question 2. Let $(S_t)_{t\in\mathbb{P}^1}$, be a pencil of quartic surfaces in $\mathbb{P}^3$, with base locus C. Assume everything is defined over a number field. There are countably many points in $\mathbb{P}^1$, all defined over a number field, where the picard number of the $K3$ surface jumps. There are thus countably many 0-cycles on C, all defined over a number field, which are obtained by restricting to C line bundles on special fibers S_t.

What is the subgroup of $CH_0(C_{\overline{\mathbb{Q}}})$ generated by these 0-cycles? What is its image under the l-adic AJ map?

NB1. I can show that it is non torsion. Over $\mathbb{C}$ for a very general pencil, it is infinitely generated even after tensoring with $\mathbb{Q}$. I do not know how to prove that it is infinitely generated in the number field situation.

NB2. These questions are related to the famous Beilinson conjecture saying that $CH_0(S_{\overline{\mathbb{Q}}})_{hom}$ should be trivial for a variety S with $q = 0$ (in our case a $K3$ surface).

Question 2. This concerns the Hodge conjecture for integral Hodge classes on abelian varieties. Let A be a ppav with Thta divisor Θ. The class $\frac{[\Theta]^{g-1}}{(g-1)!}$ is integral.

- *Is there any ppav for which this class is known to be non algebraic?*

This question is particularly interesting in the case of an intermediate Jacobian of a 3-fold X with $p_g(X) = 0$. Indeed, the question above is then a birational invariant of X.

- A related question is the following (cf [voisinjag]): Let X be a rationally connected 3-fold and $A = JX$ be its intermediate Jacobian. There is a degree 4 integral Hodge class on $X \times JX$. If this class is algebraic, it is the class of a universal codimension 2-cycle $Z \subset X \times JX$. (One has $JX \cong CH_1(X)_{hom}$).

Question 3. *Is there any example where this class is non algebraic?*

NB. This is a birationally invariant problem (relative to X), cf. [voisinjag].

Question 4. I would like to understand the skew motive of the infinite product of a $K3$ surface S. That is, consider for each k the variety S^k equipped with the projector $\frac{1}{k!}\sum_\sigma \epsilon(\sigma)\Gamma_\sigma$, where σ runs over the symmetric group $\mathfrak{S}_k$. This motive has Hodge level 2. It should be the motive of a surface.

- The generalized Hodge conjecture for the level 2 Hodge structures constructed this way is not known.

- Even assuming it, would it be possible to prove that the $K3$ surface is finite dimensional in the Kimura sense (cf. [voisinhodgebloch])?

- A final question would be to understand the Kuga-Satake construction from the point of view of this motive. (cf. [voisinsymmetric]). Indeed, the Kuga-Satake abelian variety associated to the $K3$ surface contains naturally the Deligne cohomology groups $H^3_D(\mathbb{Z}(2) = H^2(\mathbb{C})/F^2H^2(\mathbb{C}) + H^2(\mathbb{Z})$ attached to these level 2 Hodge structures.

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