

## Notes on Green functions

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We consider a reductive group  $\mathbf{G}$  over the algebraic closure  $\overline{\mathbb{F}}_q$  of a finite field  $\mathbb{F}_q$ , which is defined over  $\mathbb{F}_q$ . We are interested in the finite group  $\mathbf{G}(\mathbb{F}_q) = \mathbf{G}^F$ , where  $F$  is a *Frobenius endomorphism* corresponding to the  $\mathbb{F}_q$ -structure on  $\mathbf{G}$ .

On  $\mathbb{A}_1 = \text{Spec}(\overline{\mathbb{F}}_q[X])$ , the “geometric” Frobenius is  $X \mapsto X^q$  while the “arithmetic” Frobenius is  $\sum_i a_i X^i \mapsto \sum_i a_i^q X^i$ . The composed is the identity on points.

For  $\mathbf{G} = \text{GL}_n$ , where  $F$  raises the matrix coefficients to their  $q$ -th power, Green (1955) has defined functions attached to each rational maximal torus, such that the values of characters of  $\mathbf{G}$  at unipotent elements are given by linear combinations of such functions.

We give a survey of these functions for the case of general reductive groups, pointing out along the way which simplifications hold for  $\text{GL}_n$ .

## Cohomological induction of Deligne and Lusztig

$\mathbf{G}^F$ -classes of maximal tori are parameterized by  $H^1(F, W)$ ; given a rational maximal torus  $\mathbf{T}_w$ ,

- Deligne and Lusztig (1976) built a “cohomological induction”  $R_{\mathbf{T}_w}^{\mathbf{G}}$  which sends characters of  $\mathbf{T}_w^F$  to virtual characters of  $\mathbf{G}^F$ .
- Irreducible characters of  $\mathbf{G}^F$  appear in these virtual characters with small multiplicities independent of  $q$  (in the case of  $\mathrm{GL}_n$ , are linear combinations of such characters).
- $R_{\mathbf{T}_w}^{\mathbf{G}}$  is defined via  $\ell$ -adic cohomology: the  $\mathbf{G}^F$ -classes of pairs  $\mathbf{T} \subset \mathbf{B}$  such that  $\mathbf{T}$  is rational are parameterized by  $W$ . Given such a pair  $\mathbf{T}_w \subset \mathbf{B}$  parameterized by  $w$ , let  $\mathbf{U} = R_u(\mathbf{B})$  (so that  $\mathbf{B} = \mathbf{T}_w \cdot \mathbf{U}$ ), and let  $\tilde{\mathbf{X}}_w = \{x \in \mathbf{G} \mid x^{-1} \cdot F(x) \in \mathbf{U}\}$ .
- We have actions of  $\mathbf{G}^F$  on the left and  $\mathbf{T}_w^F$  on the right on this variety, giving rise to actions on the cohomology groups  $H_c^i(\tilde{\mathbf{X}}_w, \overline{\mathbb{Q}}_\ell)$ . For  $\theta \in \mathrm{Irr}(\mathbf{T}_w^F)$ , we define

$$R_{\mathbf{T}_w}^{\mathbf{G}}(\theta)(g) = \sum_i (-1)^i \mathrm{Trace}(g \mid H_c^i(\tilde{\mathbf{X}}_w, \overline{\mathbb{Q}}_\ell)_\theta).$$

## Unipotent characters

Though  $\tilde{\mathbf{X}}_w$  depends on  $\mathbf{B}$ , the alternating sum depends only on  $\mathbf{T}_w$  (i.e. on the class of  $w$  in  $H^1(F, W)$ ).

- For an element with Jordan decomposition  $s \cdot u$ , there is a *character formula*  

$$R_{\mathbf{T}_w}^{\mathbf{G}}(\theta)(su) = (\#C_{\mathbf{G}}(s)^{0F})^{-1} \sum_{\{h \in \mathbf{G}^F \mid h_s \in \mathbf{T}_w\}} R_{\mathbf{T}_w}^{C_{\mathbf{G}}(h_s)^{0F}}(h_u)\theta(h_s).$$
 which shows that the value is 0 if  $s$  does not have a conjugate  $\mathbf{T}_w^F$ , and otherwise depends on  $s$  only via  $\theta$  of that conjugate.
- Thus  $R_{\mathbf{T}_w}^{\mathbf{G}}(\theta)(u)$  does not depend on  $\theta$ . So, when considering values at unipotent elements, it is enough to look at the *Unipotent characters*, constituents of  $R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Id})$ .
- One has  $\langle R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Id}), R_{\mathbf{T}_{w'}}^{\mathbf{G}}(\mathrm{Id}) \rangle_{\mathbf{G}^F} = \delta_{w,w'} \#C_W(wF)$ , so  
 [to simplify I assume from now on that  $F$  acts trivially on  $W$ ]  
 one is led to introduce, for  $\chi \in \mathrm{Irr}(W)$ , the *almost characters*  
 $R_\chi := (\#W)^{-1} \sum_{w \in W} \chi(w) R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Id})$  which satisfy  
 $\langle R_\chi, R_\psi \rangle_{\mathbf{G}^F} = \delta_{\chi,\psi}$ .
- They are the irreducible unipotent characters in the case of  $\mathrm{GL}_n$ .

Classes	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$	$\begin{pmatrix} x & 0 \\ 0 & Fx \end{pmatrix}$	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$
# such classes	$q - 1$	$(q - 1)(q - 2)/2$	$q(q - 1)/2$	$q - 1$
# class	1	$q(q + 1)$	$q(q - 1)$	$q^2 - 1$
$R_{\mathbf{T}_1}^G(\alpha, \beta)$	$(q + 1)\alpha(a)\beta(a)$	$\alpha(a)\beta(b) + \alpha(b)\beta(a)$	0	$\alpha(a)\beta(a)$
$R_{\mathbf{T}_1}^G(\omega)$	$(1 - q)\omega(a)$	0	$\omega(x) + \omega(Fx)$	$\omega(a)$
$R_{\mathbf{T}_1}^G(\alpha, \alpha)$	$(1 + q)\alpha(a^2)$	$2\alpha(ab)$	0	$\alpha(a^2)$
$R_{\mathbf{T}_s}^G(\alpha \circ N_{\mathbb{F}_{q^2}/\mathbb{F}_q})$	$(1 - q)\alpha(a^2)$	0	$2\alpha(x \cdot Fx)$	$\alpha(a^2)$

in the above  $a, b \in \mathbb{F}_q^\times$ ,  $a \neq b$  and  $x \in \mathbb{F}_{q^2}^\times$ ,  $x \neq Fx$   
and  $\alpha, \beta \in \text{Irr}(\mathbb{F}_q^\times)$ ,  $\alpha \neq \beta$ ,  $\omega \in \text{Irr}(\mathbb{F}_{q^2}^\times)$ ,  $\omega \neq \omega^q$ .

## The variety $\mathbf{X}_w$

Choosing a Borel subgroup  $\mathbf{B}_0$ , the variety  $\mathcal{B}$  of Borel subgroups of  $\mathbf{G}$  can be identified to  $\mathbf{G}/\mathbf{B}_0$ .

- If  $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ , where  $\mathbf{T}$  is a maximal torus of  $\mathbf{B}_0$ , we have the *Bruhat decomposition*  $\mathbf{G} = \bigcup_{w \in W} \mathbf{B}_0 w \mathbf{B}_0$ , which implies that the orbits of  $\mathcal{B} \times \mathcal{B}$  under the diagonal action of  $\mathbf{G}$  are in bijection with  $W$ .
- We write that  $\mathbf{B} \xrightarrow{w} \mathbf{B}'$  if the orbit of  $(\mathbf{B}, \mathbf{B}')$  is parameterized by  $w$ , or equivalently  $(\mathbf{B}, \mathbf{B}') \sim_{\mathbf{G}} (\mathbf{B}_0, {}^w \mathbf{B}_0)$ .
- Then  $R_{\mathbf{T}_w}^G(\text{Id})$  is given by the cohomology of the simpler variety  $\mathbf{X}_w = \{\mathbf{B} \in \mathcal{B} \mid \mathbf{B} \xrightarrow{w} F(\mathbf{B})\}$ : we have  $R_{\mathbf{T}_w}^G(\text{Id})(g) = \sum_i (-1)^i \text{Trace}(g \mid H_c^i(\mathbf{X}_w, \overline{\mathbb{Q}}_\ell))$ .
- $\tilde{\mathbf{X}}_w$  is an étale covering of  $\mathbf{X}_w$  of group  $\mathbf{T}_w^F$ , so we could also have defined  $R_{\mathbf{T}_w}^G(\theta)$  as the cohomology of a sheaf associated to  $\theta$  on  $\mathbf{X}_w$ .

By definition the *Green function*  $Q_w(u)$  is  $R_{\mathbf{T}_w}^G(\text{Id})(u)$ .

## Parabolic induction and Lusztig induction

Let  $\mathbf{P} = \mathbf{U} \rtimes \mathbf{L}$  be a rational Levi decomposition of a parabolic subgroup of  $\mathbf{G}$ .

Then on  $\overline{\mathbb{Q}_\ell}[\mathbf{G}^F/\mathbf{U}^F]$  we have a left action of  $\mathbf{G}^F$  and a right action of  $\mathbf{L}^F$ . We define the *parabolic induction* or *Harish-Chandra induction* for  $\chi \in \text{Irr}(\mathbf{L}^F)$ :

$$R_{\mathbf{L}}^{\mathbf{G}}(\chi)(g) = \text{Trace}(g \mid \overline{\mathbb{Q}_\ell}[\mathbf{G}^F/\mathbf{U}^F]_{\chi}) = (\#\mathbf{L}^F)^{-1} \sum_{l \in \mathbf{L}^F} \chi(l) \text{Trace}((g, l) \mid \overline{\mathbb{Q}_\ell}[\mathbf{G}^F/\mathbf{U}^F]).$$

We have a generalisation of this: if  $\mathbf{L}$  is still rational but  $\mathbf{P}$  is not necessarily rational any longer, let  $\mathbf{X}_{\mathbf{U}} = \{g \in \mathbf{G} \mid g^{-1} \cdot F(g) \in \mathbf{U}\}$ . Then  $\mathbf{X}_{\mathbf{U}}$  admits an action on  $\mathbf{G}^F$  on the left and of  $\mathbf{L}^F$  on the right, and for  $\chi \in \text{Irr}(\mathbf{L}^F)$  we define

$$R_{\mathbf{L}}^{\mathbf{G}}(\chi)(g) = \sum_i (-1)^i \text{Trace}(g \mid H_c^i(\mathbf{X}_{\mathbf{U}}, \overline{\mathbb{Q}_\ell})_{\chi}).$$

The variety  $\mathbf{X}_{\mathbf{U}}$  depends on  $\mathbf{P}$ , but the alternating sum depends only on  $\mathbf{L}$ .

## The Hecke algebra and the principal series

In the case of  $\text{GL}_n$ , all the unipotent characters belong to the *principal series*, the constituents of the representation

$$R_{\mathbf{T}_1}^{\mathbf{G}}(\text{Id}) \simeq \overline{\mathbb{Q}_\ell}[\mathbf{G}^F/\mathbf{U}_0^F]_{\text{Id}} = \overline{\mathbb{Q}_\ell}[\mathbf{G}^F/\mathbf{B}_0^F] = \text{Ind}_{\mathbf{B}_0^F}^{\mathbf{G}^F} \text{Id}$$

The commuting algebra  $\text{End}_{\mathbf{G}^F}(R_{\mathbf{T}_1}^{\mathbf{G}})$  has a basis given by the operators  $T_w(\mathbf{B}) = \sum_{\mathbf{B} \xrightarrow{w} \mathbf{B}'} \mathbf{B}'$ , and is the Hecke algebra  $\mathcal{H}(W, q)$  (a deformation of the algebra of  $W$  where the order relations for the Coxeter generators  $s^2 = 1$  are replaced by quadratic relations  $(T_s + 1)(T_s - q) = 0$ ).

It can be shown that  $\mathcal{H}(W, q) \simeq \overline{\mathbb{Q}_\ell}[W]$ , and if one calls  $\chi \mapsto \chi_q$  the corresponding correspondence of characters, one gets the decomposition  $R_{\mathbf{T}_1}^{\mathbf{G}} = \sum_{\chi \in \text{Irr}(W)} \chi(1) U_{\chi}$  (for  $\text{GL}_n$  we have  $R_{\chi} = U_{\chi}$  but not in general), and we can write more generally

$$\text{Trace}(T_w \mathbf{G} \mid R_{\mathbf{T}_1}^{\mathbf{G}}) = \sum_{\chi_q \in \text{Irr}(\mathcal{H}(W, q))} \chi_q(T_w) U_{\chi}(g).$$

## Lefschetz numbers

A way of computing  $R_{T_w}^{\mathbf{G}}$  is via the *Lefschetz trace formula*: for any  $m > 0$  we have  $\sum_i (-1)^i \text{Trace}(g^{F^m} | H_c^i(\mathbf{X}_w, \overline{\mathbb{Q}}_\ell)) = \#\mathbf{X}_w^{g^{F^m}}$ .

- We want the value at  $m = 0$
- to do that we may consider the series  $f(t) = -\sum_{m=1}^{\infty} \#\mathbf{X}_w^{g^{F^m}} t^m$ , and we have  $f(\infty) = R_{T_w}^{\mathbf{G}}(g)$ .
- Indeed let  $V \subset H_c^i(\mathbf{X}_w, \overline{\mathbb{Q}}_\ell)$  be a generalized eigenspace of  $F$  for the eigenvalue  $\lambda$  and let  $\rho$  be the character of  $\mathbf{G}^F$  afforded by  $V$ : we have  $\text{Trace}(g^{F^m} | V) = \rho(g)\lambda^m$ , and  $-\sum_{m=1}^{\infty} t^m \lambda^m \rho(g) = \rho(g) \frac{-\lambda t}{1 - \lambda t}$  whose limit for  $t = \infty$  is indeed  $\rho(g)$ .

## Shintani descent

Since  $\mathbf{G}$  is connected, any element of  $\mathbf{G}$  can be written  $x^{-1}F^m(x)$ .

- If  $x^{-1}F^m(x) \in \mathbf{G}^F$ , then  $xF(x^{-1}) \in \mathbf{G}^{F^m}$  and the correspondence  $x^{-1}F^m(x) \leftrightarrow xF(x^{-1}) \in \mathbf{G}^{F^m}$  induces a well-defined map  $N_{F/F^m} : H^1(F, \mathbf{G}^F) \xrightarrow{\sim} H^1(F, \mathbf{G}^{F^m})$  ( $H^1(F, \mathbf{G}^F)$  is just the set of conjugacy classes of  $\mathbf{G}^F$ ).
- The functions on  $\mathbf{G}^{F^m}$  factoring through  $H^1(F, \mathbf{G}^{F^m})$  are generated by the restrictions to  $\mathbf{G}^{F^m} \cdot F$  of characters of  $\mathbf{G}^{F^m} \rtimes \langle F \rangle$ .
- A basis is obtained by taking the restriction of one extension  $\tilde{\rho}$  of each  $F$ -invariant character of  $\mathbf{G}^{F^m}$ .
- We have the *Shintani descent identity* (Asai, Digne-Michel 1980):

$$\begin{aligned} \#\mathbf{X}_w^{g^{F^m}} &= \text{Trace}(N_{F/F^m}(g) T_w F | R_{T_1}^{\mathbf{G}^{F^m}}) \\ &= \sum_{\chi_{q^m} \in \text{Irr}(\mathcal{H}(W, q^m))} \chi_{q^m}(T_w) \tilde{U}_\chi^{\mathbf{G}^{F^m}}(N_{F/F^m}(g) F), \end{aligned}$$

where  $\tilde{U}_\chi^{\mathbf{G}^{F^m}}$  is an extension of the irreducible character  $U_\chi^{\mathbf{G}^{F^m}}$  to  $\mathbf{G}^{F^m} \rtimes \langle F \rangle$ .

## Green functions and perverse sheaves

A difficult result, proved initially by Springer and Kazhdan (1978), then later in a different way by Lusztig and Shoji (1982), is that (for  $q$  a large enough power of a large enough prime)

$$Q_w(u) = \sum_{i \geq 0} (-1)^i \text{Trace}(wF | H^i(\mathcal{B}_u, \overline{\mathbb{Q}}_\ell))$$

where  $\mathcal{B}_u := \{\mathbf{B} \in \mathcal{B} \mid \mathbf{B} \ni u\}$ .

- One of the difficulties is to explain the action of  $W$  on the cohomology of  $\mathcal{B}_u$ . Lusztig initiated an approach using perverse sheaves.
- Let  $\mathbf{X} = \{(g, \mathbf{B}) \in \mathbf{G} \times \mathcal{B} \mid \mathbf{B} \ni g\}$ . The proper map  $\mathbf{X} \xrightarrow{\pi} \mathbf{G} : (g, \mathbf{B}) \mapsto x$  (considered by Grothendieck and Springer, in order to get a resolution of singularities of the unipotent subvariety  $\mathbf{G}_{\text{uni}}$ ) is, above the open subset  $\mathbf{G}_{\text{ss}}$  of regular semisimple elements, an étale covering of group  $W$ .

## Local systems

Thus  $\mathcal{L} := \pi_* \overline{\mathbb{Q}}_\ell$ , the direct image of the constant sheaf on  $\pi^{-1}(\mathbf{G}_{\text{ss}})$ , is a local system on  $\mathbf{G}_{\text{ss}}$  whose fibers are isomorphic to the regular representation of  $W$ .

- Given an irreducible local system  $\mathcal{L}$  on a locally closed smooth irreducible subvariety  $\mathbf{Y}$  of a variety  $\mathbf{X}$ , the intersection cohomology complex  $IC(\overline{\mathbf{Y}}, \mathcal{L})[\dim \mathbf{Y}]$  is a simple perverse sheaf, and any simple perverse sheaf on  $\mathbf{X}$  is of this form.
- We will denote  $\mathcal{L}^\#$  for  $IC(\overline{\mathbf{Y}}, \mathcal{L})[\dim \mathbf{Y}]$ .
- The action of  $W$  on  $\mathcal{L}$  gives rise to an action on  $\mathcal{L}^\#$ , and Lusztig has shown that  $\mathcal{L}^\# = \pi_* \overline{\mathbb{Q}}_\ell[\dim \mathbf{G}]$ . We get thus an action of  $W$  on  $\pi_* \overline{\mathbb{Q}}_\ell$ , and thus on the stalks  $H_x^i(\pi_* \overline{\mathbb{Q}}_\ell) = H^i(\pi^{-1}(x), \overline{\mathbb{Q}}_\ell) = H^i(\mathcal{B}_x, \overline{\mathbb{Q}}_\ell)$ .

## The Springer correspondence

Let  $\mathcal{I}$  be the set of irreducible  $\mathbf{G}$ -equivariant local systems on  $\mathbf{G}_{\text{uni}}$ ; their support is one unipotent class, so an element  $\iota \in \mathcal{I}$  is given by a pair  $(\mathcal{E}, C)$  where  $C$  is a unipotent class and  $\mathcal{E}$  is an irreducible  $\mathbf{G}$ -equivariant local system on  $C$  (which amounts to be given a character of  $A_{\mathbf{G}}(u) := C_{\mathbf{G}}(u)/C_{\mathbf{G}}^0(u)$  for  $u \in C$ ).

- Any simple  $\mathbf{G}$ -equivariant perverse sheaf on  $\mathbf{G}_{\text{uni}}$  is of the form  $\iota^{\#}$ .
- Borho and MacPherson (1981) described the decomposition in simples as well as the action of  $W$  on  $\pi_* \overline{\mathbb{Q}}_{\ell} |_{\mathbf{G}_{\text{uni}}}$ ; it is given by:  $\pi_* \overline{\mathbb{Q}}_{\ell}[\dim \mathbf{G}_{\text{uni}}] = \sum_{\chi \in \text{Irr}(W)} \chi \otimes \iota_{\chi}^{\#}$  where  $\chi \mapsto \iota_{\chi} : \text{Irr}(W) \rightarrow \mathcal{I}$  is the Springer correspondence.
- It is surjective for  $\text{GL}_n$ , but not in general (though its image contains trivial local systems on all unipotent classes). For  $\text{GL}_n$  it sends a character defined by some partition to the trivial local system on the unipotent class with the same partition.

## Characteristic functions

Let  $\mathcal{E}$  be a  $\mathbf{G}$ -equivariant local system on a locally closed  $\mathbf{G}$ -stable subvariety of  $\mathbf{G}$ .

- Assume that  $\mathcal{E}$  is  $F$ -stable, i.e. there is an isomorphism  $F^*(\mathcal{E}) \xrightarrow{\phi} \mathcal{E}$ .
- Then one may consider  $\text{Trace}(F | \mathcal{E}_x)$  for  $x \in \mathbf{G}^F$ . This defines the characteristic function  $\chi_{\mathcal{E}}$  (or  $\chi_{\phi, \mathcal{E}}$  — note that, even if  $\mathcal{E}$  is irreducible,  $\phi$  is defined only up to scalar, so we get a class function on  $\mathbf{G}^F$  defined up to a scalar).
- For a perverse sheaf  $\iota^{\#}$  with an isomorphism  $\phi : F^* \iota^{\#} \xrightarrow{\sim} \iota^{\#}$  one defines similarly  $\chi_{\phi, \iota^{\#}}(x) = \sum_i (-1)^i \text{Trace}(\phi | \mathcal{H}^i(\iota^{\#})_x)$ .
- Let  $\mathcal{Y}_i$  be the characteristic function of the local system  $\mathcal{E}$ , if  $\iota = (\mathcal{E}, C)$ ; for  $\text{GL}_n$ , it is just the characteristic function of  $C^F$ . We normalize it by setting  $\tilde{\mathcal{Y}}_i = q^{\frac{1}{2}(\dim \mathbf{G}_{\text{uni}} - \dim C)} \mathcal{Y}_i$ .
- With the above notations, the values  $R_{\chi}(u)$  (where, as above,  $R_{\chi}$  denotes the Mellin transform  $\#W^{-1} \sum_{w \in W} \chi(w) Q_w$ ), are given by  $R_{\chi}(u) = \chi_{\phi, \iota^{\#}} = \sum_{\iota} P_{\iota, \chi} \tilde{\mathcal{Y}}_{\iota}(u)$ , where  $P_{\iota, \chi}$  are polynomials in  $q$ .

## The Lusztig-Shoji algorithm (1985)

The  $P_{\iota, \iota'}$  are determined by the matrix equation:  ${}^t P \Lambda P = \Omega$  where

- $P$  is the matrix  $\{P_{\iota, \iota'}\}_{\iota, \iota'}$ . If  $\iota = (\mathcal{E}, C)$  and  $\iota' = (\mathcal{E}', C')$  then  $P_{\iota, \iota'} = 0$  unless  $C \subset \overline{C'}$ . And if  $C = C'$ , we have  $P_{\iota, \iota'} = \delta_{\mathcal{E}, \mathcal{E}'}$ . Thus, in an appropriate basis,  $P$  is unipotent lower triangular. In  $\mathrm{GL}_n$  the relation  $C \subset \overline{C'}$  is the *dominance order* on partitions; if  $C'$  corresponds to the partition  $\lambda_1 \geq \lambda_2 \geq \dots$  and  $C$  to the partition  $\mu_1 \geq \mu_2 \geq \dots$  the condition is  $\lambda_1 + \dots + \lambda_i \geq \mu_1 + \dots + \mu_i$  for all  $i$ .
- If  $\iota = \iota_\chi$  and  $\iota' = \iota_{\chi'}$ , then  $\Omega_{\iota, \iota'} = \sum_{i=0}^l q^{l-i} (-1)^i \langle \chi \otimes \chi', r^{\wedge i} \rangle_W$ , where  $r$  is the reflection representation of  $W$  and  $l$  the semi-simple rank of  $\mathbf{G}$ .
- $\Lambda_{\iota, \iota'} = \langle \mathcal{Y}_\iota, \mathcal{Y}_{\iota'} \rangle_{\mathbf{G}^F}$ . Thus, for  $\mathrm{GL}_n$  it is diagonal with diagonal coefficients  $(\#C_{\mathbf{G}^F}(u))^{-1}$  where  $u$  runs over representatives of the unipotent classes.