ARCC ABSTRACT ELEMENTARY CLASSES WORKSHOP OPEN PROBLEM SESSION

MODERATED BY A. VILLAVECES, NOTES BY M. MALLIARIS

Question 1. (Grossberg) Is there a Hanf number for amalgamation?

Let κ be a cardinal. Is there $\lambda(\mathcal{K})$ such that for any similarity type L of cardinality $\leq \kappa$ and any AEC with $LS(\mathcal{K}) \leq \kappa$ such that $L(\mathcal{K}) = L$, if $\mathcal{K}$ has λ-AP then $\mathcal{K}$ has λ^+-AP?

[Discussion] Very little is known about this question. More broadly, one can ask whether there is a general connection between AP_λ and AP_{λ^+}, though it seems more likely that there is a threshold cardinal where behavior changes.

Related questions:

- Question: Construct examples where the amalgamation property fails in small cardinals and holds everywhere above some sufficiently large λ. We know that:

  Theorem. *(Makkai-Shelah) If $\mathcal{K} = Mod(\psi)$, $\psi \in \mathcal{L}_{\kappa,\omega}$ for κ strongly compact, then $[I(\mathcal{K}, \lambda^+) = 1$ for $\lambda \geq \beth_{(2^\kappa)^+}] \to AP_\mu$ for all μ.*

  Conjecture: categoricity in sufficiently large λ implies amalgamation.

  Theorem. *(Kolman-Shelah) Like Makkai-Shelah except that κ is measurable in the assumption and in the conclusion we have AP_μ for $\mu \leq \lambda$, where λ is the categoricity cardinal.*

  Remark: in Kolman-Shelah the amalgamation property comes from a well-behaved dependence notion for $\mathcal{K} = Mod(\psi)$ (which depends on the measurable cardinal) and an analysis of limit models.

- Clarify the canonical situation: does categoricity in all uncountable cardinals imply excellence/amalgamation?
  Remark: this is known (with some set-theoretic hypotheses) for the class of atomic models of some first order theory, i.e., if $\mathcal{K}$ is categorical in all uncountable cardinals, then $\mathcal{K}^{at}$ is excellent. Another example, due to Shelah: $(2^\lambda < 2^{\lambda^+})$ If $\mathcal{K}$ is categorical in λ and λ^+, then $\mathcal{K}$ has AP_λ.

Question 2. (Scanlon) Find examples illustrating the pathologies and fine behavior of amalgamation.

More precisely, let X, Y be disjoint classes of cardinals. Find an AEC $\mathcal{K}$ which has AP_ν for $\nu \in X$, and does not have AP_μ for $\mu \in Y$.

Note that X, Y can be singletons. Possible conjecture: Either X or Y is a set.

Question 3. (Zilber) Let $\mathcal{K}$ be excellent, categorical $(+ \dots)$. What is known about ranks (or degrees) analogous to what is known in the first order context?

[Discussion] In work of Zilber, assuming categoricity and excellence of some nice constructions (universal covers of algebraic varieties) yields a list of subtle arithmetic properties which are connected to various rank notions, most interestingly the notion of height. It might be a 'rank' which controls the existence of prime models over sets.

Conjecture: We can extract in some abstract form the notion of height from the Shelah AEC ranks, i.e., going deeper into the consequences of categoricity, we might find ranks which are useful for applications in diophantine geometry.

Question: (Baldwin) Does Shelah's rank satisfy the Lascar inequalities, or is there another rank which does?

Guess: (Grossberg) In atomic excellent classes, it is probably possible to get something like Morley rank (i.e., an ordinal-valued rank such that the rank of the union of two disjoint definable sets is the max of the ranks of the sets).

Question 4. (Jarden) Recall that:

Definition. *A* <u>*superlimit model*</u> *in λ is $M \in \mathcal{K}_\lambda$ such that:*

(1) *If $\langle M_i : i < \delta \rangle$, $\delta < \lambda^+$ is an increasing continuous sequence of models, each isomorphic to M, then $\bigcup_{i<\delta} M_i$ is also isomorphic to M.*
(2) *M is universal (for $\mathcal{K}_\lambda$).*
(3) *M is not maximal.*

Question: Let X be a class of cardinals. What is the superlimit spectrum of $\mathcal{K}$? Can one find an AEC $\mathcal{K}$ such that $\mathcal{K}$ has a superlimit model of size λ iff $\lambda \in X$?

Remark: This is false in the first order case by the stability spectrum theorem. Note that if M is superlimit in $\mathcal{K}_\lambda$, we can construct a class $\mathcal{K}^*_\lambda := \{M \in \mathcal{K} : N \cong M\}$ which is also an AEC in λ.

Question: Are there trivial examples of bad behavior here? (for instance, examples where the sequences of cardinals in which superlimits do and do not exist are both cofinal)

Question 5. (Baldwin/Grossberg) Question 4 is an instance of a more general issue: What is superstability for AECs? There have been several attempts to define this:

(a) Stability from some λ on.
(b) Splitting for ω-chains of Galois types.
(c) Strong splitting for ω-chains / boundedness of rank (cf Shelah 394).
(d) Unions of saturated models are saturated.

[Discussion] Categoricity implies that (b) holds below the categoricity cardinal.

Question: (AP) Does categoricity in some cardinal above the Hanf number imply (a)? Yes for tame $\mathcal{K}$ (Grossberg-VanDieren). Without the assumption of tameness this is open.

Kesala and Hyttinen have developed a notion of superstability for finitary AECs which is similar to (b) and implies (a), for a slightly different notion of type ("weak types").

Question: (Baldwin) Assuming amalgamation and arbitrarily large models, does (a) imply [say (H_1, ∞)-] tameness? (This is another approach to the categoricity conjecture.) Also: we know that categoricity implies (H_1, ∞)-tameness. Is H_1 the best bound here?

Question 6. (VanDieren) Find examples of noncategorical AECs.

[Discussion] Find algebraic examples which will bring out the right stability notions/requirements for AECs (i.e., which will separate stability from categoricity; the hope is also that here algebraic tools would provide an alternative to the technology of Ehrenfeucht-Mostowski models). Assume JEP to avoid trivial examples.

Question: Reflect the known stability spectrum of first order theories by adding something infinitary into stable theories (for instance, exponentiation on some stable theory which is not $\aleph_1$-categorical).

Question: Let $\mathcal{K}$ be a class of [torsion/torsion-free] abelian groups. Develop a nonelementary stability theory for this class.

Question 7. (Walczak-Typke) Assuming AP but not GCH, what are the possible relationships between the universality spectrum and the stability spectrum?

[Discussion – Walczak-Typke] In the first-order context, universality spectrum can be viewed as an alternate means for the classification of theories, particularly for unstable theories (see Shelah, Lazy Model Theorists' Guide). In this context, universality spectra are calculated assuming failure of GCH (otherwise all complete FO theories have universal models in all cardinals). Assuming the failure of relevant instances of GCH, more information is yielded. In particular, for T a complete theory and $\lambda \geq |T|$, if T is λ-stable, then T has a universal model of cardinality λ. In the unstable case, one can look at whether it is possible to force a universal family to be small, using appropriate forcing (i.e., make the smallest number of models of size λ needed to embed all other models of size λ to be $< 2^\lambda$). Roughly speaking, a FO theory that can be forced to have a small universal family is called *amenable* (see DzSh 710). To give some examples, it has been found that countable simple theories are amenable for $\lambda = \kappa^+$ with $\kappa^{<\kappa}$, while theories with the strict order property and theories with SOP_4 are not.

How much of this can be extended to the context of AECs, assuming AP and the failure of GCH? Or even assuming GCH? How about the Excellent context?

Problems for Small Group Discussion

Moderated by M. VanDieren

(1) (Scanlon) Classify the $\aleph_0$-stable abstract elementary classes of fields. (In the first-order case, this is Macintyre's theorem.)

(2) (Calvert) Translate AEC concepts (ranks, etc) into the context of an appropriate class of abelian groups. (Baldwin has examples of non-tameness and amalgamation in this context.)

(3) (Grossberg) Does the boundedness of rank in 394 imply stability?

(4) (Kesala) Remove the uncountable cofinality assumption from the $\underline{\text{a}}$-categoricity cardinal in the finitary proof of $\underline{\text{a}}$-categoricity implies superstability.

(5) (Kesala) In a simple, $\aleph_0$-stable finitary class, is the cardinality of the set of Lascar strong types over a finite set always countable? (This is related to questions about the existence of an $\aleph_0$-stable nontame finitary class.)

(6) (Baldwin) Does stability in the categoricity cardinal imply (H_1, λ)-tame? (Baldwin-VanDieren have shown the converse.)

(7) (Walczak-Typke) For λ singular, in some nonelementary context, does λ-stability imply the existence of a universal model?

(8) (Goodrick) Does there exist a dimension theory for indiscernible sequences in some AECs with a nice orthogonality notion?

(9) (Malliaris) Give a finer proof of Shelah's presentation theorem for $\aleph_0$-Galois-stable classes.