Tropical Geometry and Affine Buildings

Jenia Tevelev (with Sean Keel)

July 14, 2007

1 Tropical algebraic geometry

Consider the pictures in Figure 1. The objects are roughly the same in dimension 1 (see [1]). We will discuss the case of dimension 2 and higher in this lecture. This is joint work with Keel [2].

Let X be an algebraic variety. There are two main ways in which the tropicalization $\mathcal{T}(X)$ arises—as the degeneration of X or as the compactification of X.

Suppose X is over $\mathbb{C}$, and let $K = \mathbb{C}((z))$ be the field of Puiseux series in z with coefficients in $\mathbb{C}$. Then $X(K)$ degenerates to a compactification $\overline{X}(\mathbb{C})$ under $\gamma(z) \mapsto \gamma(0)$. On the other hand, there is a map from the set of valuations of the function field $k(X)$ to $\overline{X}$ sending a valuation v to the center of v, which is a scheme theoretic point in $\overline{X}$.

Let $\mathcal{O}^*(X)$ be the invertible rational functions on X and k the function field of X. Then $M = \mathcal{O}^*(X)/k^*$ is an integer lattice, isomorphic to some $\mathbb{Z}^r$. The tropical variety $\mathcal{T}(X)$ lives in $\mathrm{Hom}(M, \mathbb{Q})$.

Definition (Non-archimedean amoeba definition)**.** The tropical variety $\mathcal{T}(X)$ is the set $\{[\gamma] : \gamma \in X(K)\}$, where $[\gamma](m) := \deg[m(\gamma(z))]$ for $m \in \mathcal{O}^*(X)$.

Definition (Divisorial definition)**.** The tropical variety $\mathcal{T}(X)$ is the set

$$\{[v] : v \text{ is a divisorial valuation of } \mathbb{C}(X)\},$$

where $[v](m) := v(m)$ for $m \in \mathcal{O}^*(X)$.

Suppose $X \subset \overline{X}$ is such that $\overline{X}$ is smooth, $\overline{X}\backslash X$ has normal crossings, and $\overline{X}\backslash X = D_1 \cup \cdots \cup D_s$ are divisors, where $[D_i] = [\mathrm{val}_{D_i}] \in \mathcal{T}(X)$. Then

$$\mathcal{T}(X) = \bigcup_{D_{i_1}\cap\cdots\cap D_{i_s}\neq\emptyset} \mathbb{Q}_{\geq 0}[D_{i_1}] + \cdots + \mathbb{Q}_{\geq 0}[D_{i_s}].$$

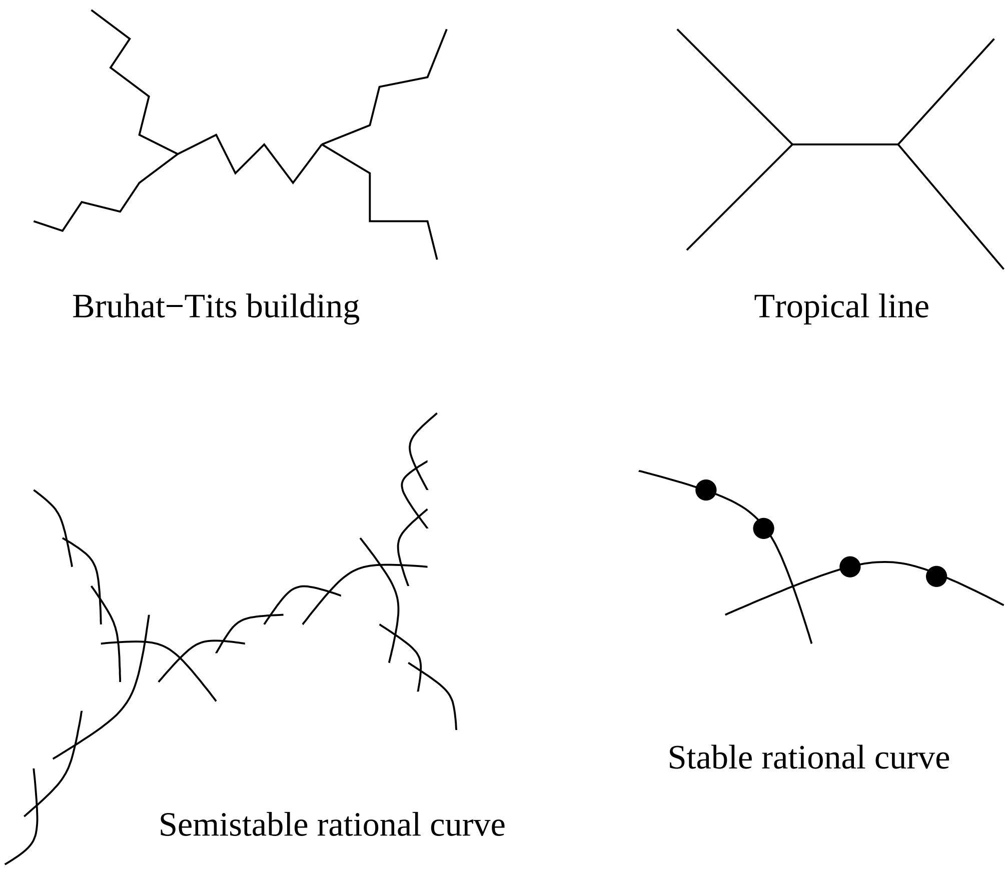

Figure 1: A Bruhat-Tits building, a tropical line, a semistable rational curve, and a stable rational curve.

Examples.

1. Let $X = \mathbb{P}^1 \backslash \{a_1, \ldots, a_{n-1}, \infty\}$. Then $\overline{X} = \mathbb{P}^1$, and $\mathcal{O}^*(X)/\mathbb{C}^* = \{x - a_1, x - a_2, \ldots, x - a_{n-1}\}$, and

   $$[a_1] = (1, 0, \ldots, 0), \ldots, [a_{n-1}] = (0, 0, \ldots, 1), [\infty] = (-1, -1, \ldots, -1).$$

   The tropical variety consists of n rays in the irections $e_1, e_2, \ldots, e_{n-1}, -(e_1 + \cdots + e_n)$.

2. Let $X = \mathbb{P}^2 \backslash \{L_1, \ldots, L_n\}$, where $L_1, \ldots, L_n$ are lines in general linear position. Then $\mathbb{P}^2$ is a compactification of X with normal crossing divisor boundary. The tropical variety $\mathcal{T}(X)$ is a two-dimensional fan on the rays $[L_1], \ldots, [L_n]$, and every pair of rays spans a cone in $\mathcal{T}(X)$ because every two lines intersect in $\mathbb{P}^2$.

   Now suppose $L_1 \cap L_2 \cap L_3 = \{p\} \in \mathbb{P}^2$. Then we can blow up $\mathbb{P}^2$ at p to obtain a compactification of X with normal crossing divisor boundary. After blowing up, the lines L_1, L_2, and L_3 no longer intersect, but each meets the exceptional divisor E. So we have to replace the three cones $\mathbb{R}_{\geq 0}[L_1] + \mathbb{R}_{\geq 0}[L_2]$, $\mathbb{R}_{\geq 0}[L_1] + \mathbb{R}_{\geq 0}[L_3]$, and $\mathbb{R}_{\geq 0}[L_2] + \mathbb{R}_{\geq 0}[L_3]$ from above with three new cones of the form $\mathbb{R}_{\geq 0}[E] + \mathbb{R}_{\geq 0}[L_i]$ for $i = 1, 2, 3$.

Let $R = \mathbb{C}[[z]]$. Then X_R is smooth and proper over Spec R, and the complement has normal crossings:

$$\begin{array}{ccc} X_K & \hookrightarrow & X_R \\ \downarrow & & \downarrow \\ \text{Spec } K & \hookrightarrow & \text{Spec} R \end{array}$$

Let $\widetilde{M} = \mathcal{O}^*(X)/R^*$. For each boundary divisor D, compute $[D] \in \widetilde{N}$, define the fan the same way, and intersect with the fiber for the map $\widetilde{N} \to \mathbb{Z}$.

Example. Let $X = \mathbb{P}^1 \backslash \{0, 1, z, \infty\}$. Then $\mathcal{O}^*(X)/R^*$ is generated by $x, x-1, x-z$, and z. The divisors $0, z$, and the special fiber over 0 intersect, so we blow up to get an exceptional divisor D_2 and the pullback D_1 of the special fiber:

$$[0] = (1,0,0,0), \qquad [1] = (0,1,0,0), \qquad [z] = (0,0,1,0),$$
$$[\infty] = (-1,-1,-1,0), \qquad [D_1] = (0,0,0,1), \qquad [D_2] = (1,0,1,1).$$

Now the question is: How do we compute normal crossing models? We can compute limits of lines.

$$\begin{array}{ccc} \mathbb{P}^2_K & \hookrightarrow & \mathbb{P}^2_R \\ \downarrow & & \downarrow \\ \text{Spec } K & \hookrightarrow & \text{Spec } R \end{array}$$

Embeddings $\mathbb{P}^2_K \hookrightarrow \mathbb{P}^2_R$ are classified by the affine Grassmannian $\mathcal{B} = \mathrm{PGL}_3(K)/\mathrm{PGL}_3(R)$. An element $[\Lambda] \in \mathcal{B}$ is an equivalence class of a lattice $\Lambda \in K^3$, where two lattices Λ_1, Λ_2 are equivalent if $\Lambda_1 = z^a \Lambda_2$ for some integer a. This gives an arrangement of lines $\mathbb{P}^2_{R,\Lambda}$ in $\mathbb{P}^2_R$. Let C_Λ be the limiting configuration of lines.

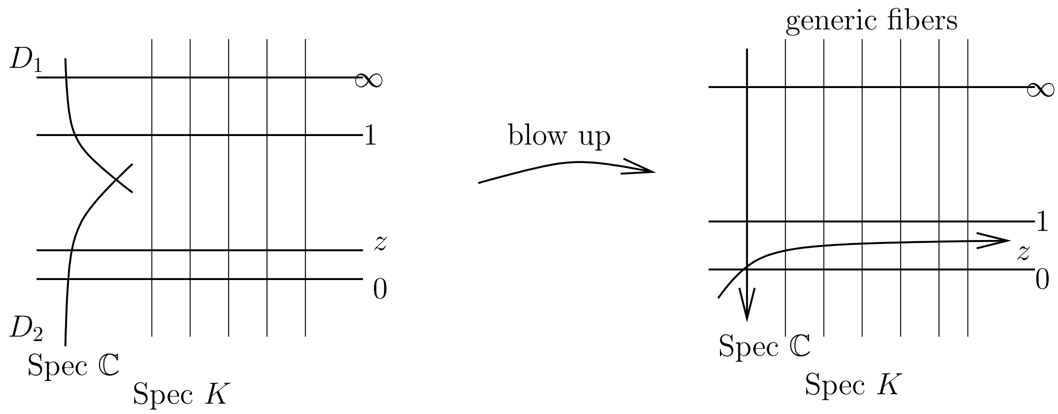

Figure 2: Blowing up $\mathbb{P}^2$.

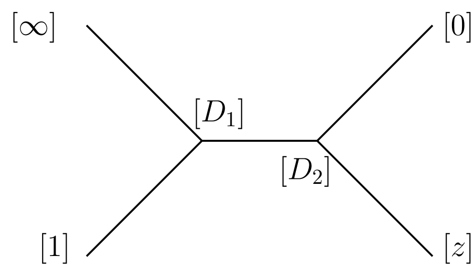

Figure 3: A tropical line.

Definition. A membrane $[M] \subset \mathcal{B}$ is the set of $[\Lambda] \in \mathcal{B}$ such that C_Λ contains a triangle of lines.

What does a membrane look like?

The compactified building $\overline{\mathcal{B}} \supset \mathcal{B}$ is the set of all free R-modules (of any rank) in K^3, up to equivalence. Let $f_1, \dots, f_n \in K^3$ such that the hyperplane L_i is the vanishing set of f_i. Then $Rf_1, \dots, Rf_n$ are on the boundary of $\overline{\mathcal{B}}$.

Lemma. The membrane M is the tropical convex hull of $Rf_1, \dots, Rf_n$. In fact, M is the union of apartments generated by linearly independent $f_{i_1}, f_{i_2}, f_{i_3}$.

Theorem. We have

$$\mathbb{P}^2_K \hookrightarrow \prod_{\Lambda \in M} \mathbb{P}^2_R.$$

Let S be the closure with divisors $L_1, \dots, L_m$, smooth with normal crossing boundary.

Theorem. The membrane is homeomorphic to the tropical 2-plane, and the homeomorphism is given by

$$\begin{aligned} \Psi \quad &: \mathcal{B} \longrightarrow \mathbb{Q}^n/(1,1,\dots,1) \\ &\Lambda \longmapsto (\min\{a_1 : z^{a_1} f_1 \in \Lambda\}, \dots, \min\{a_n : z^{a_n} f_n \in \Lambda\}) \end{aligned}$$

The same works in general for $\mathbb{P}^m$ for any m.

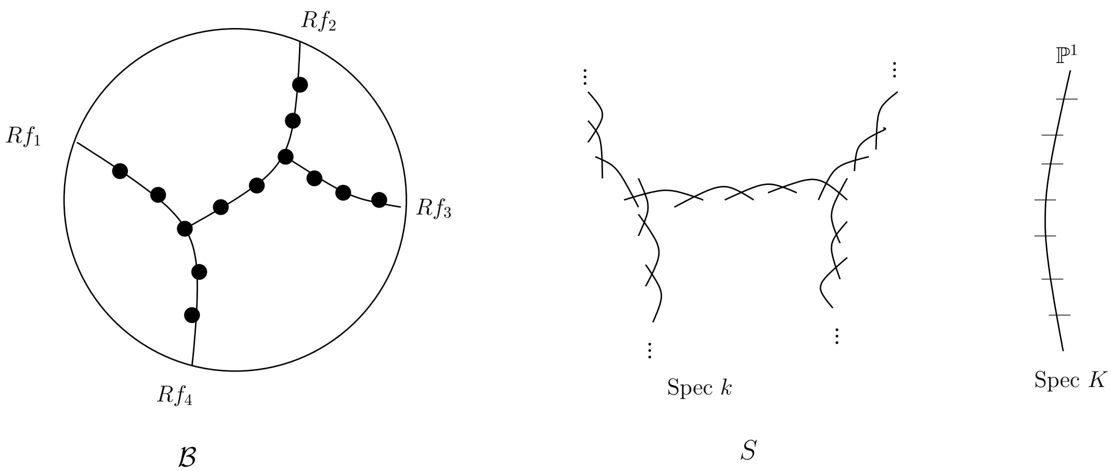

Figure 4: Compactified affined building and compactification of X.

References

[1] MIKHAIL KAPRANOV: *Veronese curves and Grothendieck-Knudsen moduli space M(0,n)*, *Journal of Algebraic Geometry* **2** (1993), 239–262

[2] SEAN KEEL AND JENIA TEVELEV: *Geometry of Chow Quotients of Grassmannians*, *Duke Math. J.* **134**, no. 2 (2006), 259–311