Langlands

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Our goal is to extend our work down the sequence

$$G \text{ reductive group}/\mathbb{C} \quad \overset{\text{problem}}{\rightsquigarrow} \quad \text{numbers, rings, } \ldots, \text{ categories}$$

We want to describe numbers, rings, etc. in terms of combinatorics. A solution should be a Chevalley description such as root data.

The geometric Satake is a geometric construction of the duual group $G^\vee$.
Consider the affine Grassmannian

$$\mathrm{Gr}_G = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]]) =: G(K)/G(\mathcal{O}),$$

an infinite-dimensional variety. Note that $G(\mathbb{C}((t)))$ is the loop group and $G(\mathbb{C}[[t]])$ is the arc group.
We have the Cartan decomposition

$$G(\mathcal{O})\backslash \mathrm{Gr}_G \simeq \Lambda_G/W_G \simeq \Lambda_G^+ \overset{!}{\simeq} \check{\Lambda}_{G^\vee}^+,$$

where $\simeq$ denotes equivalence, Λ_G^+ is a set of dominant weights for G, and $\check{\Lambda}_{G^\vee}^+$ is a set of dominant weights for the dual group $G^\vee$.
The combinatorics of the representation theory of $G^\vee$, denoted $\mathrm{Rep}(G^\vee)$, is encoded in

$$\begin{array}{rcl} \text{convolution} & \longleftrightarrow & \otimes \\ \text{MV cycles} & \longleftrightarrow & \text{weight multiplicities} \end{array}$$

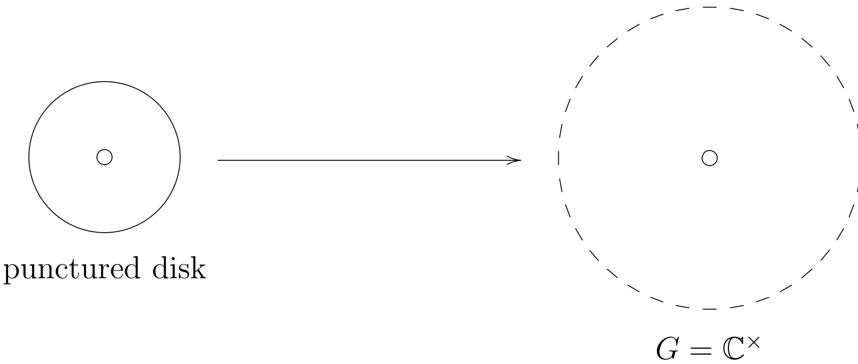
All of this can be encoded as a categorical statement—namely, that the perverse sheaves $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ are equivalent, at least as abelian categories, to the representations of $G^\vee$ $\mathrm{Rep}(G^\vee)$:

$$\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G) \simeq \mathrm{Rep}(G^\vee)$$

as abelian categories.
What goes in the ellipses above are abelian categories.

Example. Let $G = \mathrm{GL}_1$.

Then $\mathrm{Gr}_G \cong \mathbb{Z}$ as a topological space.



Since we only need to worry about the order of the pole,

$$\cdots \quad \bullet \quad \underset{-1}{\bullet} \quad \underset{0}{\bullet} \quad \underset{1}{\bullet} \quad \underset{2}{\bullet} \quad \underset{3}{\bullet} \quad \bullet \quad \cdots$$

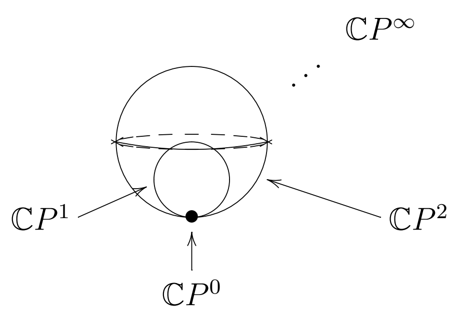
we assign a (finite-dimensional) vector space to each point.

$$\mathrm{Perv}_{\mathrm{GL}_1(\mathcal{O})}(\mathrm{Gr}_G) \simeq \mathbb{Z} - \text{graded vector spaces} \overset{!}{\simeq} \mathrm{Rep}(G)$$

But we're supposed to be doing $G(\mathcal{O})$-equivariant geometry on Gr_G. In other words, we should be doing geometry on $G(\mathcal{O})\backslash\mathrm{Gr}_G$ (note that as a set, $G(\mathcal{O})\backslash\mathrm{Gr}_G$ is Λ_G^+).
What should replace the $\mathbb{Z}$-graded vector spaces?
For GL_1, it should be $\mathrm{GL}_1(\mathcal{O})\backslash\mathbb{Z}$ (stack, groupoid, "Artin orbifold," ...).
To a topologist who has never met an algebraic geometer, $\mathrm{GL}_1(\mathcal{O}) = \mathbb{C}^\times \simeq S^1$, so

$$\mathrm{GL}_1(\mathcal{O})\backslash\bullet^0 \quad = \quad S^1\backslash\bullet \quad \simeq \quad S^1\backslash S^\infty \quad = \quad \mathbb{C}P^\infty$$

What is $\mathbb{C}P^\infty$?



We replace $\mathbb{Z}$ with

$$\cdots \quad \overset{\mathbb{C}P^\infty}{\bullet} \quad \overset{\mathbb{C}P^\infty}{\bullet} \quad \overset{\mathbb{C}P^\infty}{\bullet} \quad \overset{\mathbb{C}P^\infty}{\bullet} \quad \overset{\mathbb{C}P^\infty}{\bullet} \quad \cdots$$

A vector space is a $\mathbb{C}$-module. We had

$$\bullet \rightsquigarrow \text{algebra } \mathbb{C} = H^*(\mathrm{pt}) \rightsquigarrow \text{category of } \mathbb{C}\text{-modules}$$

In the new picture,

$$\mathbb{C}P^\infty \rightsquigarrow H^*(\mathbb{C}P^\infty) = \mathbb{C}[u] \rightsquigarrow \begin{array}{c}\text{category of differential}\\ \text{graded } \mathbb{C}[u]\text{-modules}\end{array}$$
$$\deg u = 2$$

We thus have a new Satake category: differential graded (dg) $\mathbb{C}[u]$-modules (two gradings). What is the spectral description of this? In other words, what should we consider instead of $\mathrm{Rep}(\mathrm{GL}_1)$?
Earlier, we had $\mathrm{Rep}(\mathrm{GL}_1)$.
But now we have $\mathbf{\Lambda} = \mathbb{C} \oplus \mathbb{C} \cdot \lambda_{-1}\,(= H_*\,(S^1))$, where $\deg \lambda_{-1} = -1$ and $\lambda_{-1}^2 = 0$.
Put $\mathbb{C}[u] =: \mathbb{S}$.
graded (dg) $\mathbf{\Lambda}$-modules

$$\Big(\cdot\Big) \quad \Big(\cdot\Big) \quad \Big(\cdot\Big) \quad \Big(\cdot\Big) \quad \Big(\cdot\Big)$$

The "derived" geometric Satake

$$\begin{aligned} \text{graded (dg) } \mathbf{\Lambda}\text{-modules} &\simeq \text{ graded (dg) } \mathbb{S}\text{-modules} \\ \mathbf{\Lambda} &\mapsto \text{ trivial module} \\ \text{trivial module} &\mapsfrom \mathbb{S} \end{aligned}$$

Note that the above equivalence is equivalence of categories.

In general (Drinfeld: "The Exercise"):

$$D_{G(\mathcal{O})}(\mathrm{Gr}_G) \simeq \; G^\vee\text{-equivariant dg } \Lambda(\mathfrak{g}^\vee[1])\text{-modules},$$

where $D_{G(\mathcal{O})}(\mathrm{Gr}_G)$ is the set of all $G(\mathcal{O})$-equivariant sheaves on the Grassmannian.

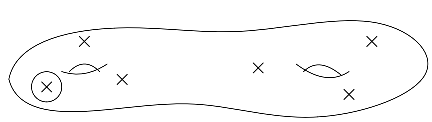
Remarks.

1. This is a local statement.
2. The degree of the set of all $G^\vee$-equivariant dg $\Lambda(\mathfrak{g}^\vee[1])$-modules is -1.
3. In the case of GL_1,
   (a) $D_{G(\mathcal{O})}(\mathrm{Gr}_G)$ is $\mathbb{S}$,
   (b) the set of all $G^\vee$-equivariant dg $\Lambda(\mathfrak{g}^\vee[1])$-modules is $\mathbf{\Lambda}$.

Where is $\mathbf{\Lambda}$ coming from?
The role of the geometric Satake in the geometric Langlands:

geometric Langlands → harmonic analysis on locally symmetric spaces
geometric Langlands → homological mirror symmetry (topological field theory)

Locally symmetric space: Let C be a Riemann surface



and $\mathrm{Bun}_G(C)$ the space of all principle G-bundles on C (when $G = \mathrm{GL}_1$, then $\mathrm{Bun}_G(C)$ is the space of all line bundles on C).
Adelic description (Weil): remove enough points so that the G-bundle is trivial.

$$\prod_{p \in C} G(\mathcal{O}_p) \Big\backslash \prod_{p \in C} G(K_p) \Big/ G(\mathbb{C}(C)) = \mathrm{Bun}_G(C)$$

Remarks.

1. $G(K_p)$ is the loop group at p
2. $\mathrm{Bun}_G(C)$ is a global object

In terms of points, the above is

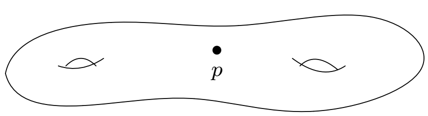
gauge on disks\gluing data/gauge on C generic.

Now we want to do harmonic analysis.
Let the operator $K\backslash G/K$ act on $K\backslash G/\Gamma$.
Then for each $p \in C$,

$$G(\mathcal{O}_p)\backslash G(K_p)/G(\mathcal{O}_p).$$

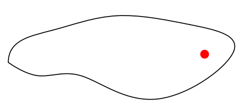
What is the conjectured spectral space for $\mathrm{Bun}_G(C)$ with the Satake action?

$G(\mathcal{O}_p)\backslash G(K_p)/G(\mathcal{O}_p)$



eigenvalue: $\mathrm{Rep}(G^\vee) \longrightarrow$ vector spaces
$V \longmapsto \mathcal{P} \times_{G^\vee} V$
($\mathcal{P}$ a principle $G^\vee$-bundle on the point)

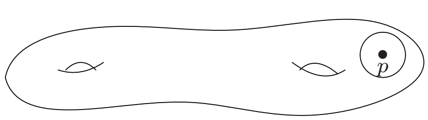
The spectral space is $\mathrm{Conn}_{G^\vee}(C)$, the set of all $G^\vee$-connections on C.



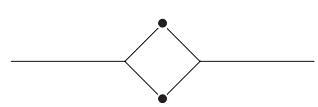
potential eigenvalue

$\mathrm{Conn}_{G^\vee}(C)$

Finally, what is the $\mathrm{Rep}(G^\vee)$-action on this side? Also, where does $\mathbf{\Lambda}$ for GL_1 come from?
Recall: The Satake action on $\mathrm{Bun}_G(\mathbb{C})$ at a point $p \in C$ was "regluing."
Let's mimic this for connections.

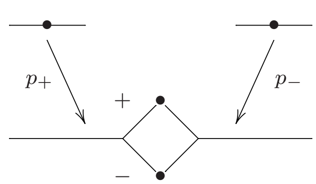


$\mathbb{C}$



$\mathbb{D}$ double punctured disk

Consider $\mathrm{Conn}_{G^\vee}(\mathbb{D})$.



$$
\begin{tikzcd}
\mathrm{Conn}_{G^\vee}(D) & & \mathrm{Conn}_{G^\vee}(D) \\
& \mathrm{Conn}_{G^\vee}(\mathbb{D}) \arrow[ul, "p_+^*"] \arrow[ur, "p_-^*"'] &
\end{tikzcd}
$$

Thus,

$$\mathrm{Conn}_{G^\vee}(\mathbb{D}) = \{0\} \bigcap_{\mathfrak{g}^\vee} \{0\} = \mathrm{Spec}\left(\mathbb{C}_0 \bigotimes_{\mathrm{Sym}(\mathfrak{g})}^{\text{derived}} \mathbb{C}_0 \right) = \mathrm{Spec}\left(\mathbf{\Lambda}^{-*}\ \mathfrak{g}^\vee \right),$$

where $\{0\} \cap_{\mathfrak{g}^\vee} \{0\}$ is the trivial connection and $\mathrm{Sym}(\mathfrak{g})$ denotes the symmetric algebra.

Bridgeland: bases for categories, mutations