Models for Crystals

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Includes joint work with A. Postnikov (MIT).
Papers: `arXiv:math`, `math.albany.edu/math/pers/lenart`

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- character formulas;
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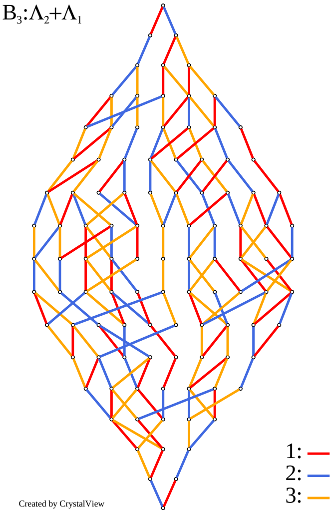
- branching rules;
- description of Lusztig's involution (to be mentioned).

A crystal



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I will present the alcove path model (L. and Postnikov).

Weyl group:

$$W = \langle s_\alpha \ : \ \alpha \in \Phi \rangle = \langle s_i \ : \ i = 1, \ldots, r \rangle .$$

Length: $\ell(w) = \min \{ k \ : \ w = s_{i_1} \ldots s_{i_k} \} .$

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Bruhat graph: directed graph on W with labeled edges

$$w \xrightarrow{\alpha} ws_\alpha \ \text{ if } \ \ell(ws_\alpha) = \ell(w) + 1 \, .$$

Alcoves

Hyperplanes $H_{\alpha,k} = \{\lambda : \langle \lambda, \alpha^\vee \rangle = k\}$ ($k \in \mathbb{Z}$).

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Fundamental alcove:

$$A_\circ = \{\lambda \in V \,:\, 0 < \langle\lambda, \alpha^\vee\rangle < 1 \text{ for } \alpha \in \Phi^+\} .$$

Given $\lambda \in \Lambda^+$, let

$$(A_\circ = A_0, A_1, \ldots, A_l = A_\circ - \lambda)$$

be a shortest sequence of adjacent alcoves (alcove path).

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Indexing set $\mathcal{A}(\lambda) = \mathcal{A}(\lambda, \Gamma)$ for a basis of V_λ; consists of subsets

$$J = \{j_1 < j_2 < \ldots < j_s\} \subseteq \{1, \ldots, l\}$$

such that we have the following path in the Bruhat graph:

$$1 \xrightarrow{\beta_{j_1}} w_1 \xrightarrow{\beta_{j_2}} w_2 \ldots \xrightarrow{\beta_{j_s}} w_s =: \kappa(J) \;\; \text{(key)}\,.$$

Such subsets will be called admissible subsets.

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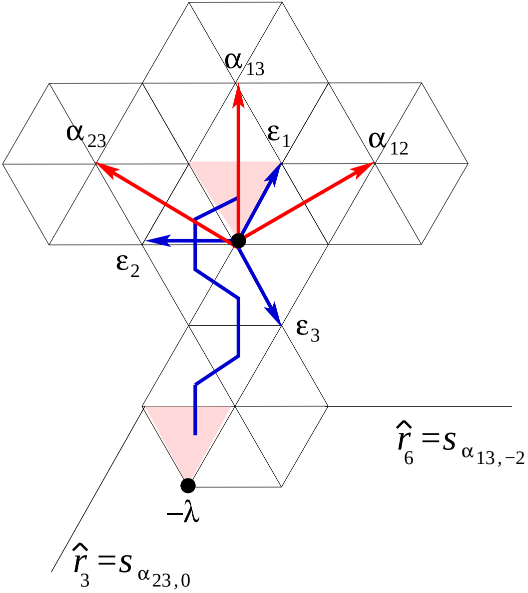
Weight of an admissible subset:

$$\mu(J) := -\widehat{r}_{j_1} \ldots \widehat{r}_{j_s}(-\lambda).$$

Example. Type A_2, $\lambda = 3\varepsilon_1 + \varepsilon_2$.



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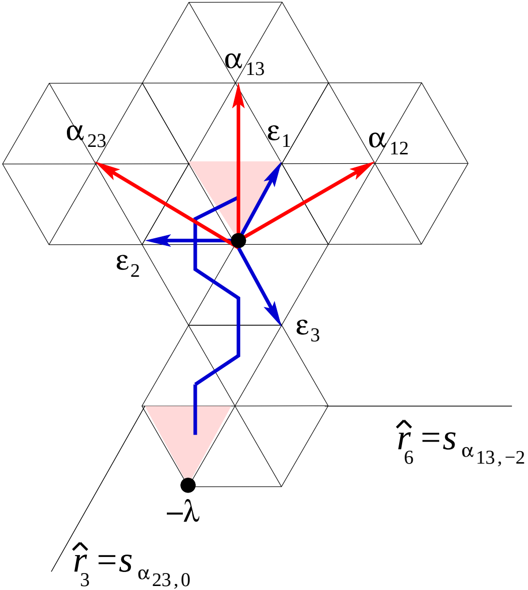

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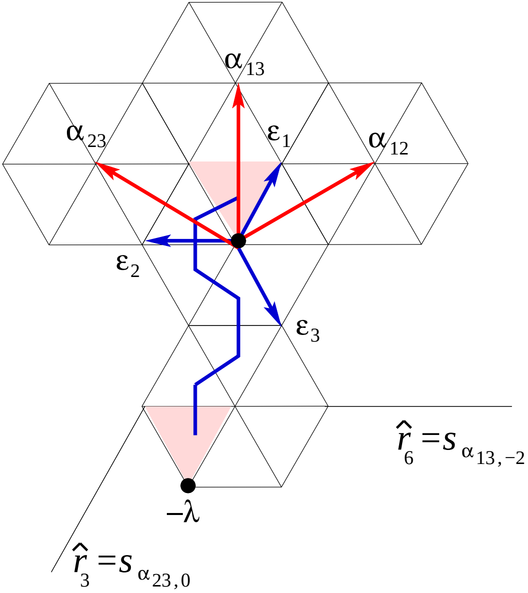

$J = \{3, 6\}$, saturated chain $e = 123 < t_{23} = 132 < t_{23}t_{13} = 231$.
$J = \{6\}$ not admissible: $e < t_{13} = 321$.

Theorem. (L. and Postnikov) *The irreducible character* $ch(V_\lambda)$ *of* $\mathfrak{g}$ *can be expressed as*

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Remark. There is a similar Demazure character formula.

Crystal graphs

Theorem. (L. and Postnikov) *The crystal graph structure corresponding to* V_λ *can be defined combinatorially on* $\mathcal{A}(\lambda)$ *by directed edges*

$$J \mapsto (J \setminus \{m\}) \cup \{k\}\,.$$

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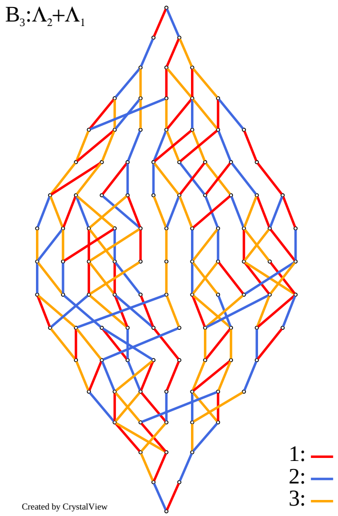
$$J \mapsto (J \setminus \{m\}) \cup \{k\}\,.$$

There is a corresponding poset structure on $\mathcal{A}(\lambda)$. Minimum $J_{\text{min}} = \emptyset$ and maximum J_{max}.



Created with P. Sternberg's "CrystalView".

Fact. (Lusztig) $\mathcal{A}(\lambda)$ *is a* *self-dual* *poset, i.e. there is a bijection* $\eta : \mathcal{A}(\lambda) \to \mathcal{A}(\lambda)$ *such that*

$$J \leq J' \iff \eta(J) \geq \eta(J').$$

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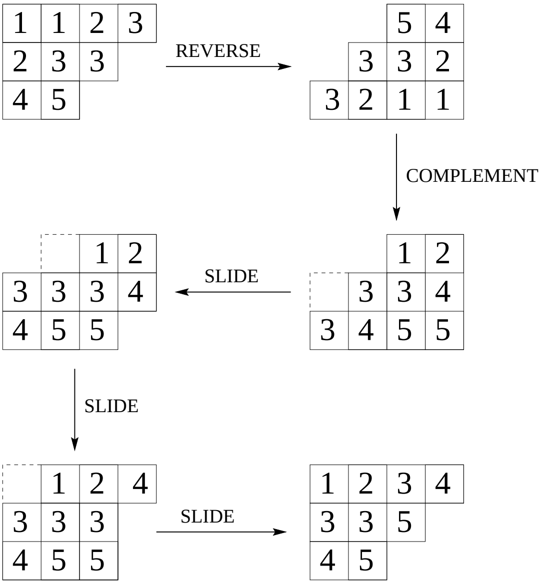
In type A, it is given by Schützenberger's evacuation on semistandard Young tableaux (Berenstein and Zelevinsky).

Schützenberger's evacuation



Generalizing Schützenberger's evacuation

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STEP 1 (REVERSE-COMPLEMENT)
Define a bijection

$$J \in \mathcal{A}(\lambda, \Gamma) \;\mapsto\; J^{\mathrm{rev}} \in \mathcal{A}(\lambda, \Gamma^{\mathrm{rev}})\,,$$

such that

$$\mu(J^{\mathrm{rev}}) = w_\circ(\mu(J))\,.$$

Example.

Type A_2, $\lambda = 4\varepsilon_1 + 2\varepsilon_2$, $J = \{2, 4\}$,

$$\begin{array}{cccccccc} \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 \\ \Gamma = (\alpha_{12}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & \underline{\alpha_{12}}, & \alpha_{13}, & \underline{\alpha_{23}}, & \alpha_{13}) \end{array}$$

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$J^{\mathrm{rev}} = \{\overline{1}, 2, 4\}$

STEP 2 (SLIDE)

Yang-Baxter moves. Let Γ, Γ' be λ-chains related as follows:

$$\Gamma = (\beta_1, \ldots, (\beta_i, \beta_{i+1}, \ldots, \beta_j), \ldots \beta_l) \ \mapsto$$
$$\Gamma' = (\beta_1, \ldots, (\beta_j, \beta_{j-1}, \ldots, \beta_i), \ldots \beta_l)\,,$$

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Theorem. (L.) *There is a bijection*

$$J \in \mathcal{A}(\lambda, \Gamma) \overset{YB}{\mapsto} J' \in \mathcal{A}(\lambda, \Gamma')$$

$$\textit{such that} \quad J \setminus [i,j] = J' \setminus [i,j], \quad \kappa(J) = \kappa(J'), \quad \mu(J) = \mu(J').$$

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Theorem. (L.) *We have* $J^* = \eta(J)$.

Example.

$J^{\mathrm{rev}} = \{\overline{1},\ 2,\ 4\}$

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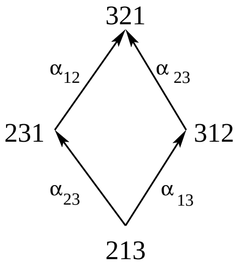

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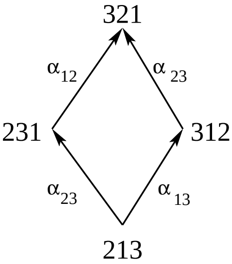

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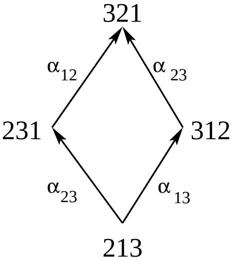
Example.

$J^{\mathrm{rev}} = \{\overline{1},\, 2,\, 4\}$

$$\begin{array}{rcccccccc} & \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 \\ \Gamma^{\mathrm{rev}} = & (\underline{\alpha_{12}}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & (\underline{\alpha_{23}}, & \alpha_{13}, & \underline{\alpha_{12}}), & \alpha_{13}) \end{array}$$

$$\begin{array}{rcccccccc} & \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 \\ \Gamma = & (\ \underline{\alpha_{12}}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & (\alpha_{12}, & \underline{\alpha_{13}}, & \underline{\alpha_{23}}), & \alpha_{13}) \end{array}$$



$J = \{2, 4\} \mapsto J^* = \{\overline{1}, 3, 4\}$

Idea of proof: Show that the map $J \mapsto J^*$ commutes with the directed edges of the crystal graphs as required.