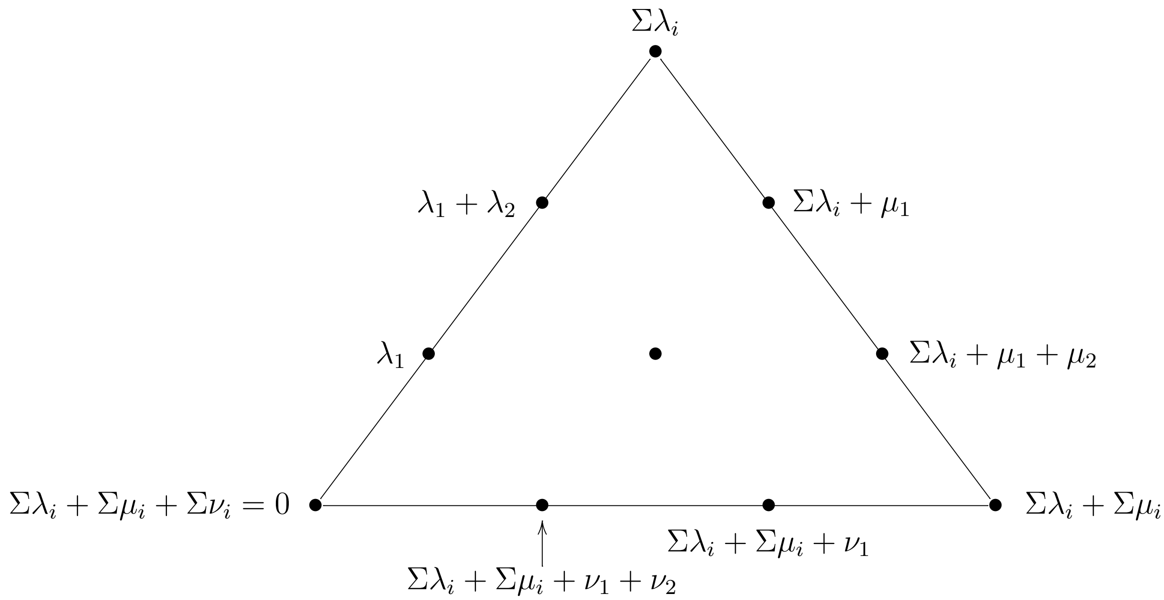
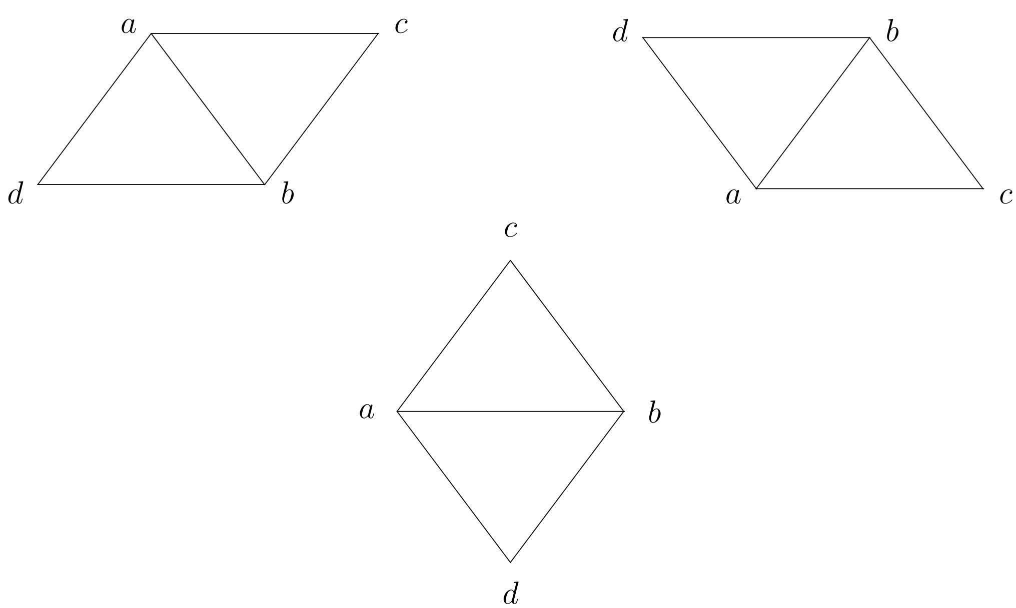
Honeycombs

Allen Knutson (with Terry Tao)

Definition (Johnson, 1970; Berenstein-Zelevinsky, 1993)**.** A *hive* for GL_n is an $(n+1)$-size triangle ($n+1$ dots on each side of the triangle) of real numbers or integers



such that for all rhombi



$a + b \geq c + d$ (convexity of hive mound). These inequalities are "rhombus inequalities," and $a + b - c - d$ is a "rhombus functional."

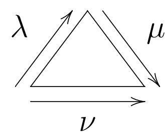
Note that convexity implies $\lambda_1 \geq \cdots \geq \lambda_n$, $\mu_1 \geq \cdots \geq \mu_n$, and $\nu_1 \geq \cdots \geq \nu_n$.

Theorem (Berenstein-Zelevinsky)**.** The number of integer hives with boundary data (λ, μ, ν) is $\dim(V_\lambda \otimes V_\mu \otimes V_\nu)^{\mathrm{GL}_n}$. Moreover,

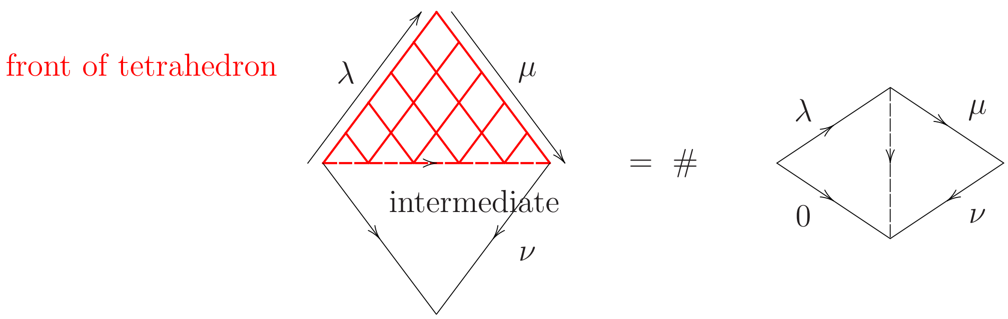
$$(\lambda, \mu, -\underbrace{w_0}_{\text{reverse}} \cdot \nu) = \dim(\mathrm{Hom}_{\mathrm{GL}_n}(V_\nu, V_\lambda \otimes V_\mu) = c^\nu_{\lambda\mu}.$$

Definition. A *hive* (not necessarily associative) *ring* Hive_n has $\mathbb{Z}$-basis $\{b_\lambda : \lambda = (\lambda_1 \geq \cdots \geq \lambda_n)\}$ and

$$b_\lambda \cdot b_\mu := \sum_\nu b_\nu \cdot \# \text{ hives}$$



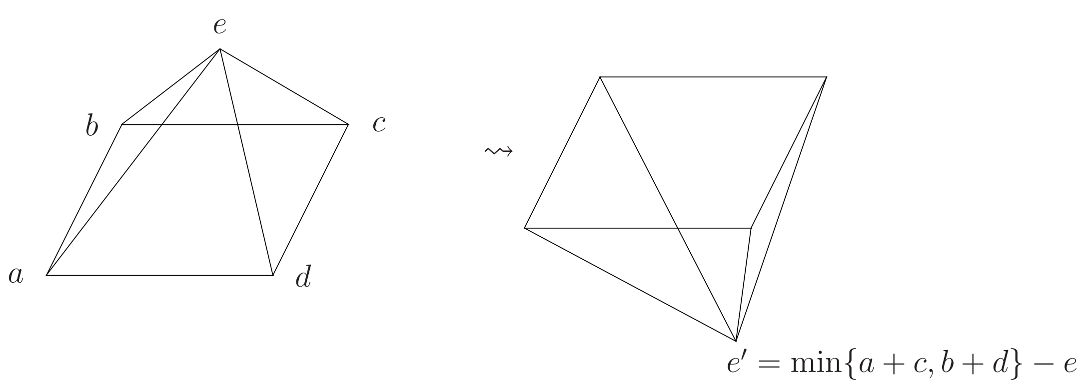
This has an obvious bijection with $\mathrm{Rep}(\mathrm{GL}_n(\mathbb{C}))$ via $b_\lambda \mapsto [V_\lambda]$.
(joint with Chris Woodward) To show this is a ring homomorphism (and hence an isomorphism), it is enough to prove associativity and the Pieri formula $\lambda = (1, 1, \ldots, 1, 0, \ldots, 0)$.
For associativity, consider



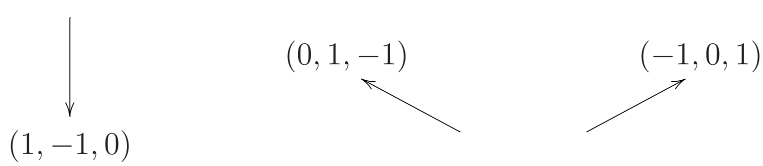
Then

$$\mathrm{Hom}_{\mathrm{GL}_n}(V_0, (V_\lambda \otimes V_\mu) \otimes V_\nu) \cong \mathrm{Hom}_{\mathrm{GL}_n}(V_0, V_\lambda \otimes (V_\mu \otimes V_\nu)).$$

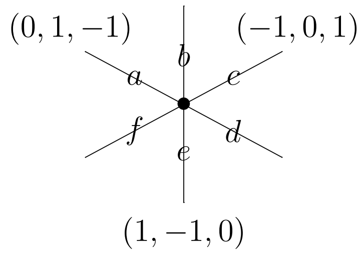
The rule ("octahedron recurrence") for labelling new lattice points in the interior upon excavating an octahedron is as follows:



Definition. A *honeycomb* is a picture in $R^3_{\Sigma=0}$ (the sum of the coordinates is 0) that is a union of finitely many line segments (sometimes semi-infinite line segments) with $\mathbb{N}_+$-multiplicities all parallel to

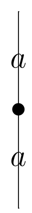


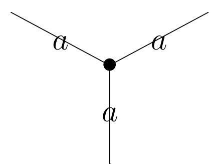
Each edge has a “constant coordinate.” There is a local condition called “zero tension”: The total vector sum with multiplicites must be $\vec{0}$:

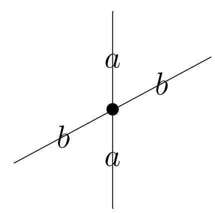


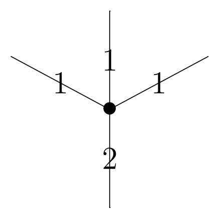
$$(-b-c+e+f, a+b-d-e, -a+c+d-f) = (0,0,0)$$

Example.

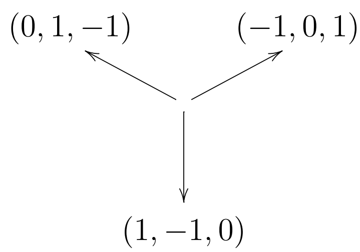




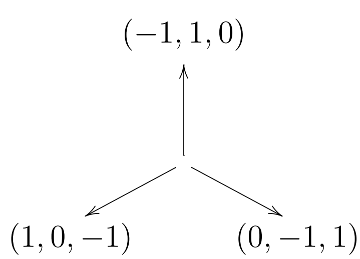




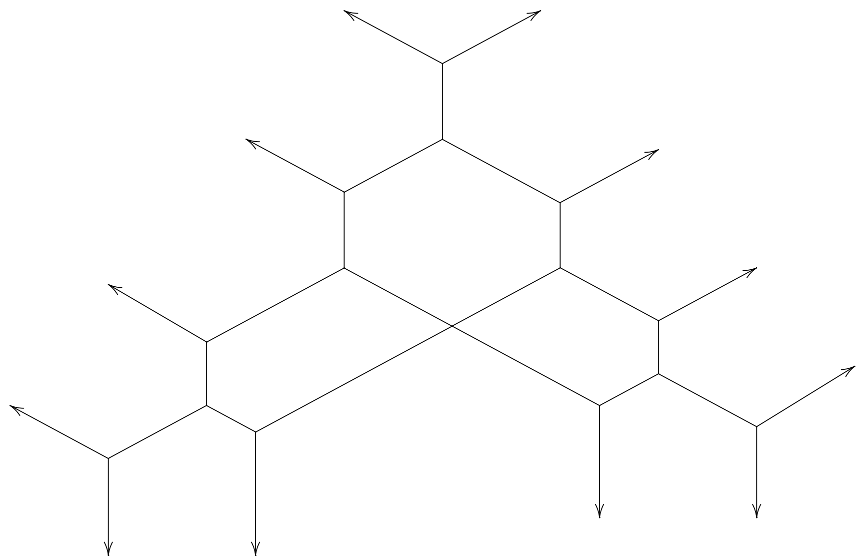
There is also a global condition: The semi-infinite line segments only go in the directions

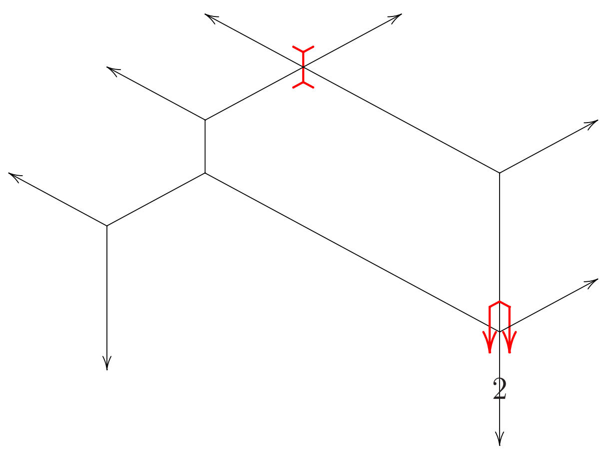


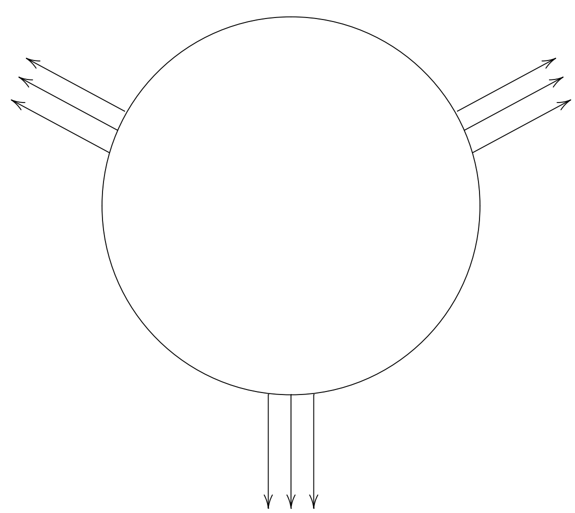
not



Example. All the line segments in the following honeycomb have multiplicity 1.



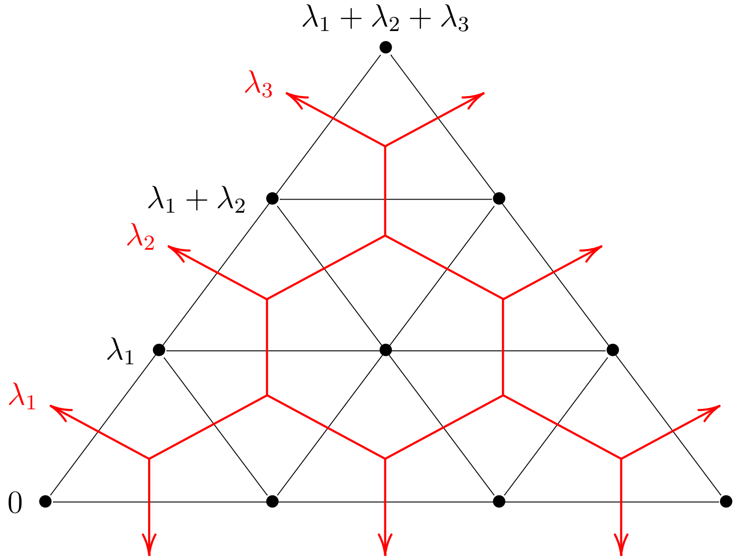




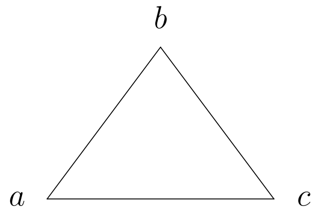
same number (with multiplicity) of semi-infinite line segments in the three directions

hives	$\twoheadrightarrow$	honeycombs
rhombus inequalities	$\rightsquigarrow$	edges have non-negative length

Example.



honeycomb is dual graph to hive



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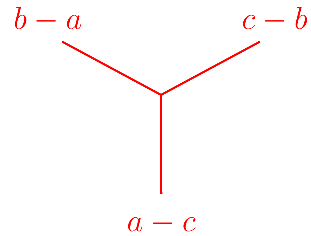

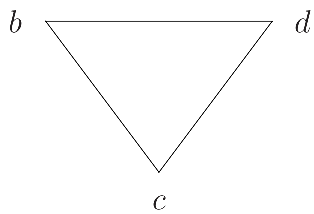

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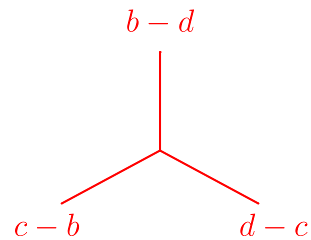

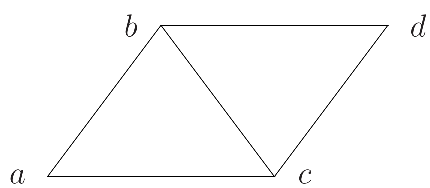

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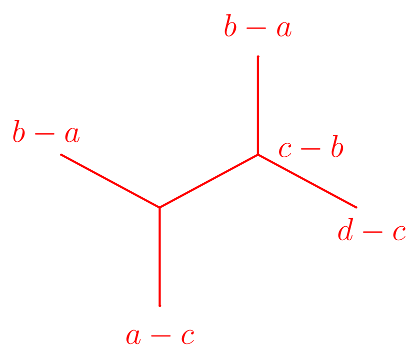


Theorem. This is a bijection.

What inequalities are true of λ, μ, ν arising as the boundary data of a real hive?
Obvious inequalities ("chamber inequalities"): $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, and $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$
An example of a more interesting inequality is $\lambda_1 + \mu_1 \geq -\nu_n$.
To study these inequalities, we can assume that $\lambda_1 > \lambda_2 > \cdots > \lambda_n$, $\mu_1 > \mu_2 > \cdots > \mu_n$, and $\nu_1 > \nu_2 > \cdots > \nu_n$; i.e., λ, μ, ν are *regular* dominant.

Definition. Hives_n is the set of all size n real hives.

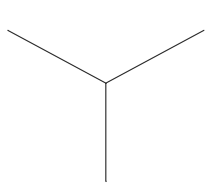
Remarks.

1. The dimension of Hives_n is $\binom{n+1}{2} - 1$.

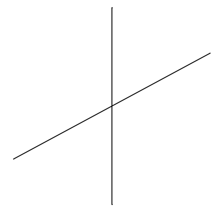
2. By taking the boundary, $\mathrm{Hives}_n \twoheadrightarrow C$, and C has dimension $3n - 1$.

Theorem. There is a continuous, piecewise linear section $C \to \mathrm{Honey}_n$ called the "largest lift" such that if λ, μ, ν are regular dominant and $(\lambda, \mu, \nu) \in C$, then the largest lift of (λ, μ, ν)

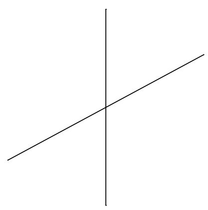
(a honeycomb) has no multiplicities. Up to rotation, all vertices look like



or



and if we regard



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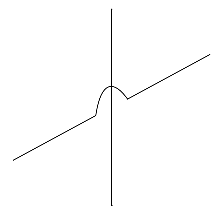
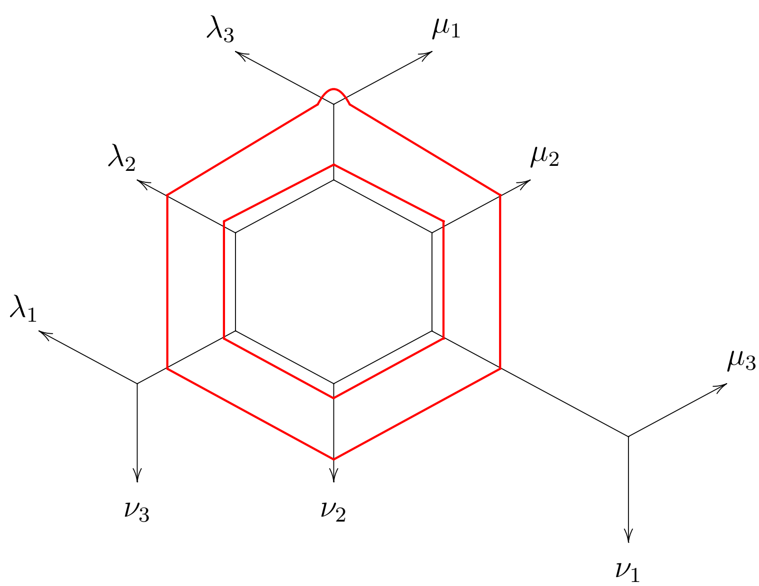
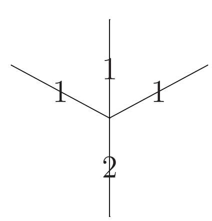


then the honeycomb is acyclic.

Example. This has a cycle but is not a largest lift.

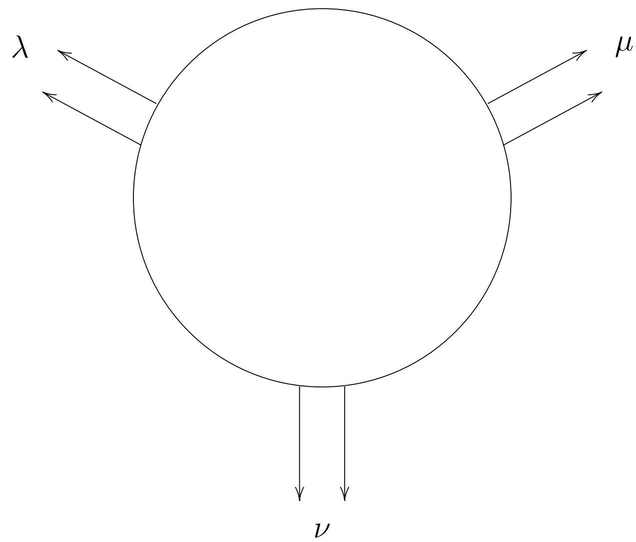


Once we have a cycle, we can use functionals to maximize. Here's the idea: the largest lift hive (associated to an element of C) is defined to be the hive that maximizes a certain linear functional. If the largest lift hive has a cycle, we can "expand" it (the cycle) and increase the functional; hence, the largest lift hive can have no cycles. It's harder if we have



Corollary. Saturation.

Proof. This "largest lift" is defined with $\mathbb{Z}$-coefficients. Thus, λ, μ, ν are integral, and there is an integer honeycomb.



$\square$