# Overview of connections between buildings and representation theory, and open problems

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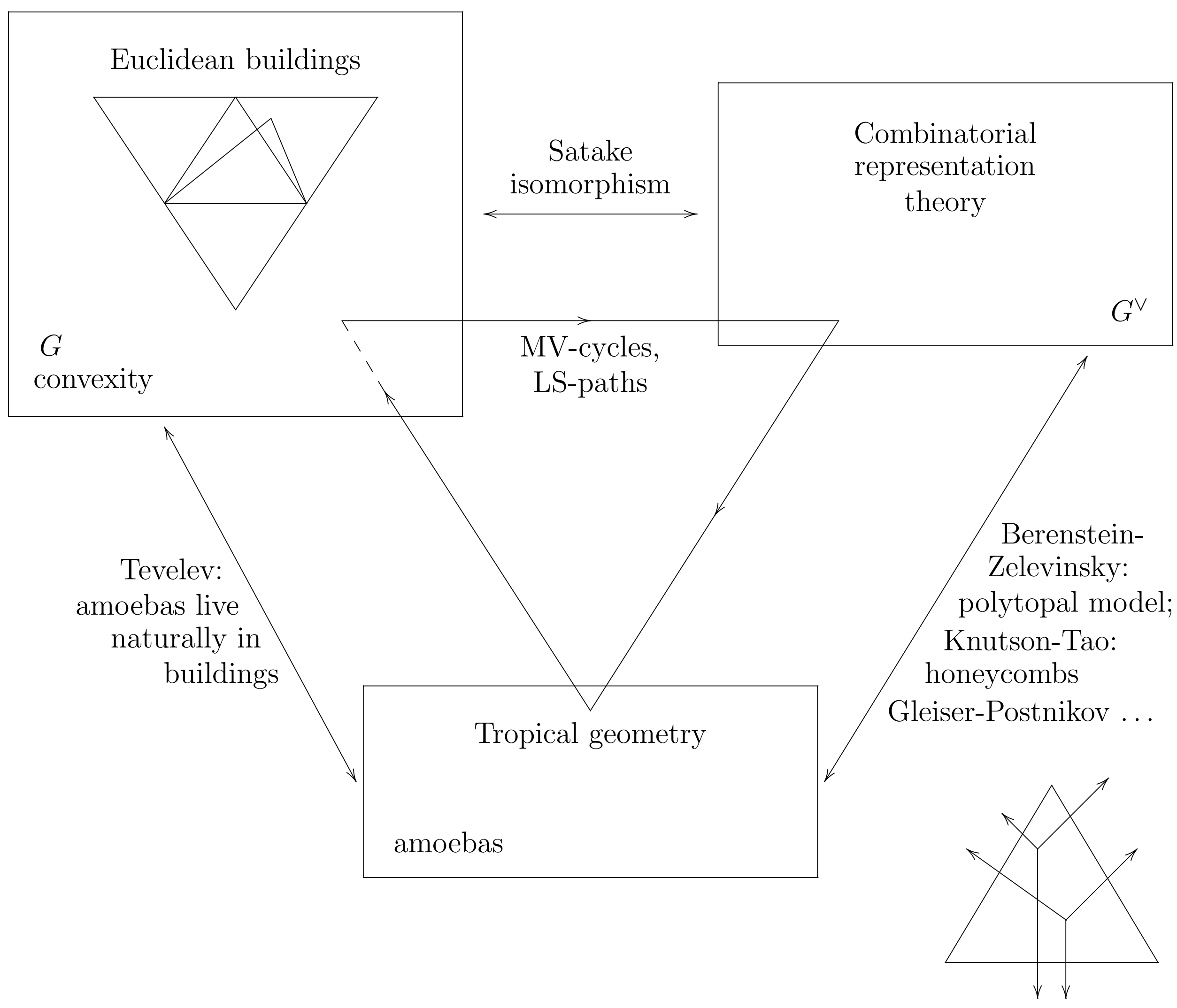
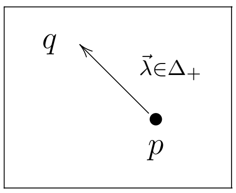
Why are we here? The goal of this workshop is to walk along the perimeter of the following triangle:

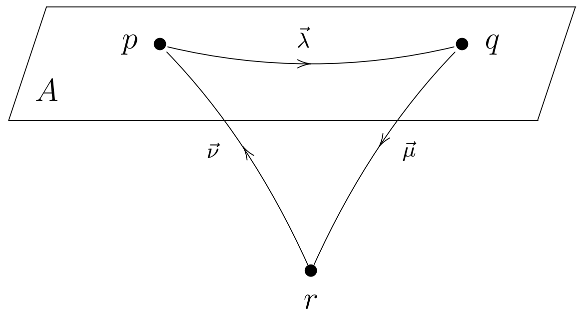



It appears that convexity is a common theme in both "building" and "tropical" approaches to the representation theory.

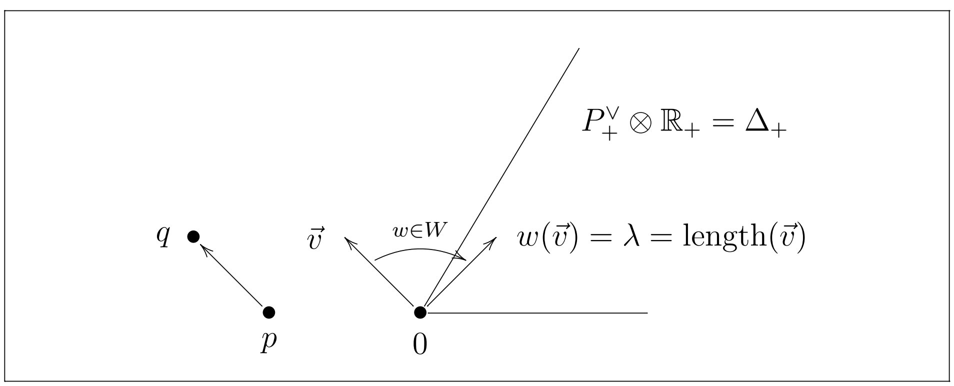
In this talk I will mostly concentrate on the edge of the triangle connecting buildings and representation theory. Connections between representation theory and tropical geometry will be explored in the talks by Berenstein and Knutson. Connections between tropical geometry and buildings are very recent: It will be discussed in the talk of Tevelev.

Geometric questions about Euclidean buildings:
We will think of a Euclidean building as the geometric realization of an abstract simplicial complex. This way we have well-defined concepts of straight lines and lengths of geodesics.




$\vec{\lambda}, \vec{\mu}, \vec{\nu}$ are vectors in $\Delta_+ = P_+^\vee \otimes \mathbb{R}_+$ and represent side-lengths of a triangle in the Euclidean building.
Definition of $\Delta_+$-valued distance between points in the Euclidean building.

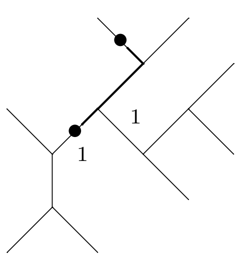



$$\vec{d}(p,q) = \vec{\lambda}$$

(move $\vec{pq}$ to origin and rotate)

**Question.** What are the triangle inequalities in this geometry? I.e., what are the restrictions on the side-length vectors of a triangle in the Euclidean building. Note that, a priori, it's not clear that these restrictions are given by inequalities.

**Example.** In the case of $\mathrm{SL}_2(\mathbb{Q}_p)$, the building is a regular tree.




$\Delta_+ = \mathbb{R}_+$

triangle inequalities: the usual triangle inequality

It turns out that there is a system of linear homogeneous inequalities with integer coefficients (triangle inequalities) TI on $(\lambda, \mu, \nu) \in \Delta \times \Delta \times \Delta$ that are necessary and sufficient conditions for there to be a triangle with side-lengths $\lambda, \mu, \nu$ that answer the above question.

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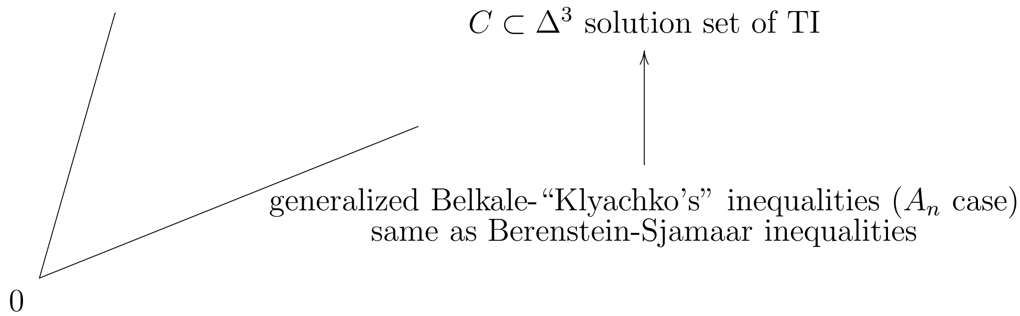
Key tool for the proof: Analyzing certain convex functions on Euclidean and spherical buildings.




**Question.** Irredundancy of the triangle inequalities?

In $A_n$ case: Knutson-Tao-Woodward proved irredundancy of TI

In $B_n, C_n$ cases: $C_{B_n} = C_{C_n}$ (cones—solution sets—are the same)
In particular, $C$ depends only on $W$, but $\mathrm{TI}_{B_n} \neq \mathrm{TI}_{C_n}$ (inequalities redundant)
$\mathrm{TI}_{B_n}$ are Belkale-Klyachko/Berenstein-Sjamaar inequalities
However there is a smaller system of inequalities $\mathrm{TI}^{\mathrm{BK}}$—Belkale-Kumar inequalities—smaller, which also describe the cone $C$.

**Conjecture** (Belkale–Kumar)**.** $\mathrm{TI}^{\mathrm{BK}}$ are irredundant.

The inequalities TI come from Schubert calculus over $G(\mathbb{C})/P(\mathbb{C})$, where $P(\mathbb{C})$ are maximal parabolic subgroups. The inequalities TI have the following form:

$$\langle \lambda, w_\lambda(\varpi_i) \rangle + \langle \mu, w_\mu(\varpi_i) \rangle + \langle \nu, w_\nu(\varpi_i) \rangle \leq 0.$$

Here $\varpi_i, i = 1, \ldots, \ell$ are the fundamental weights and $w_\lambda, w_\mu, w_\nu \in W$ are triples such that

$$[C_{w_\lambda}] \cdot [C_{w_\mu}] \cdot [C_{w_\nu}] = [point],$$

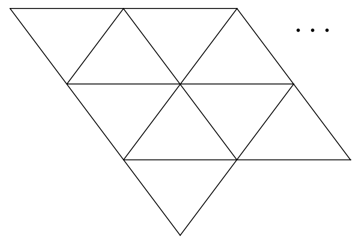
where $C_w$ is the Schubert cycle in the Grassmanian $\mathrm{Gr}_{\varpi_i} = G(\mathbb{C})/P_{\varpi_i}(\mathbb{C})$ corresponding to the coset $wW_{\varpi_i} \in W/W_{\varpi_i}$. The product is the intersection product in the integer homology ring $H_*(\mathrm{Gr}_{\varpi_i}, \mathbb{Z})$.
Belkale-Kumar inequalities have the same form except that they consider a deformed product in $H_*(\mathrm{Gr}_{\varpi_i}, \mathbb{Z})$.

Connection to Hecke Algebra: We now impose certain integrality conditions on the triangles in Euclidean buildings.

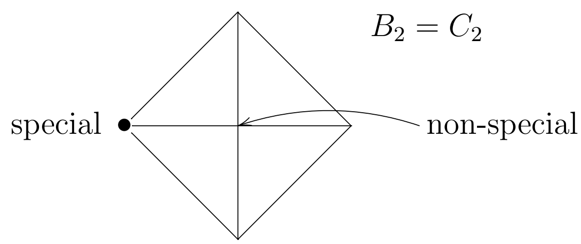
1. Consider $\lambda, \mu, \nu \in P_+^\vee$.

2. Moreover, consider only triangles in the Euclidean building whose vertices are special vertices of the building.

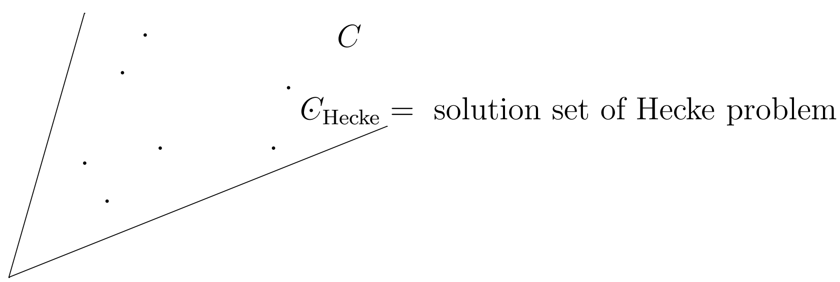
$A_n$ case: all vertices special



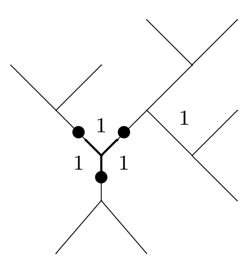
point is <u>special</u> if its stabilizer in $\widetilde{W}$ is isomorphic to $W$




**Problem** (Restricted triangle/Hecke problem)**.** What are the restrictions on the side-lengths $\lambda, \mu, \nu \in P_+^\vee$ so that there exists a triangle in the Euclidean building with special vertices and side-lengths $\lambda, \mu, \nu$?




Obvious necessary condition: $\lambda + \mu + \nu \in Q^\vee \subset P^\vee$ (congruence condition)
Reason for the obvious necessary condition




$\lambda = \mu = \nu = 1$
$1 + 1 + 1 = 3 \notin 2\mathbb{Z} = Q^\vee$

It turns out that $(\lambda, \mu, \nu) \in C_{\text{Hecke}}$ iff

$$f_\lambda * f_\mu * f_\nu = \sum c_{\lambda\mu\nu}(\delta) \cdot f_\delta, \qquad \text{(where } c_{\lambda\mu\nu}(0) \neq 0 \text{ with 0 the trivial character).}$$

Here $f_\lambda, f_\mu, f_\nu$ are $K$-bi-invariant functions on $G_{\mathbb{Q}_p}$ and $*$ denotes the convolution product. Moreover, $c_{\lambda\mu\nu}(0) = \#[\text{triangles (appropriately normalized)}]/G$.

**Problem** (Representation Theory problem Rep for $G^\vee$)**.** Find necessary and sufficient conditions on $\lambda, \mu, \nu$ such that

$$(V(\lambda) \otimes V(\mu) \otimes V(\nu))^{G^\vee} \neq \{0\},$$

where $V(\gamma)$ is the irreducible representation of $G^\vee(\mathbb{C})$ with the highest weights $\gamma$.

Let $C_{\text{Rep}}$ denote the solution set of this problem.
It turns out that (Kapovich, Leeb, Millson):

$$C_{\text{Rep}} \subset C_{\text{Hecke}} \subset C$$

**Remark.** $C_{\text{Rep}}$ and $C_{\text{Hecke}}$ are both discrete, but $C$ is continuous.

For the $A_n$ root system ($\mathrm{SL}_{n+1}$ case), the sets are "the same":

$$C_{\text{Hecke}}(A_n) = C_{\text{Rep}}(A_n) = C \cap (P^\vee)^3 \cap \{\lambda + \mu + \nu \in Q^\vee\}$$

Requiring that $(\lambda, \mu, \nu)$ belongs to the lattice

$$L = (P^\vee)^3 \cap \{\lambda + \mu + \nu \in Q^\vee\}$$

is called the integrality condition.
The first equality above was proved in the 1960s by Hall and Klein, and later reproved by Kapovich-Millson (buildings), Haines (algebraic geometry). The second equality was proven by Knutson-Tao (through "tropical geometry"), Derksen-Wyman (algebraic geometry), Kapovich and Millson (buildings) and Belkale (algebraic geometry).
The equality $C_{\text{Rep}} = C \cap (P^\vee)^3$ is equivalent to the saturation conjecture.

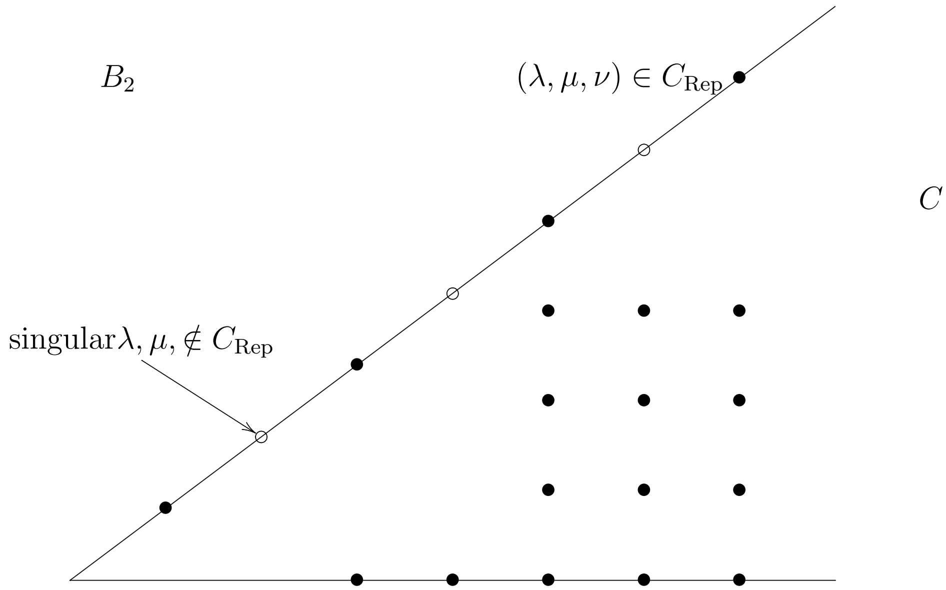
For other root systems, have (sometimes) strict inequalities:
For all non-simply laced root systems we have:

$$C_{\text{Rep}} \subsetneqq C_{\text{Hecke}} \subsetneqq C \cap \text{ integrality condition}$$

**Remark.** While $C_{\text{Rep}}$ is a semigroup, $C_{\text{Hecke}}$ is not a semigroup in general $(B_2, G_2)$.

**Conjecture** (Generalized saturation conjecture)**.** Equalities hold for all simply-laced root systems. Equivalently, saturation holds for all simply-laced root systems.

**Example.** Below is the picture of how $C_{\text{Rep}}$ looks like for $B_2$:




For $B_2$, the semigroups $C \cap$ integrality conditions and $C_{\text{Rep}}$ differ only on the faces of the cone $C$, where two of the three dominant weights are multiples of $\varpi_2$. On these faces, $(\lambda, \mu, \nu) \in C_{\text{Rep}}$ if and only if $\lambda + \mu + \nu \in 2P^\vee$.

**Example.** Root system $G_2$.
Pick a ray $\rho = \mathbb{R}_+ \cdot \delta$ in the cone $C$, where $\delta$ is a primitive vector in the lattice

$$L := \{(\lambda, \mu, \nu) \in P^\vee : \lambda + \mu + \nu \in Q^\vee\}.$$

Then either

$$\rho \cap L = C_{\text{Rep}} \cap L$$

or

$$\rho \cap L = \{\delta\} \cup C_{\text{Rep}} \cap L.$$

Partial evidence for the generalized saturation conjecture: for $A_n, D_4$,

$$C_{\text{Rep}} = C_{\text{Hecke}} = C \cap \text{ integrality conditions}$$

$D_4$ was done by a massive computer computation. Plus computer experiments for other simply-laced root systems.

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**Conjecture** (in the non-simply laced case (Knutson-Tao))**.** Suppose $(\lambda, \mu, \nu) \in C$ is so that $(\lambda + \mu + \nu)(t) = 1$ for all $t \in T$ such that $Z_{G(\mathbb{C})}(t)$ is semisimple. Here $T$ is the maximal torus of $G(\mathbb{C})$ and $Z(t)$ denotes the centralizer of $t$. Then

$$(\lambda, \mu, \nu) \in C_{\text{Rep}}.$$

**Conjecture** (Kapovich, Millson)**.** For every root system, the intersection of $C_{\text{Rep}}$ with the interior of the cone $C$ is saturated. Moreover, if $\lambda, \mu, \nu$ are regular, then

$$(\lambda, \mu, \nu) \in C_{\text{Rep}} \iff (\lambda, \mu, \nu) \in C \cap L;$$

i.e.,

$$C_{\text{Rep}} \cap (\text{regular}) = C \cap (\text{regular}).$$

**Theorem** (Kapovich, Leeb, Millson)**.**

$$k_R \cdot C_{\text{Hecke}} \subset C_{\text{Rep}},$$
$$k_R \cdot C \cap L \subset C_{\text{Hecke}},$$

where $k_R = \operatorname{lcm}(a_1, \ldots, a_\ell)$ when the highest root for $G^\vee$ has the form $\theta = \sum_{i=1}^{\ell} a_i\alpha_i$.