Conjectures and Open Problems

This document contains conjectures and questions that were either mentioned during the workshop or included in participants' contributions.

Contents

1 Some Open Problems and Partial Solutions from the Final Report

The information contained here is excerpted from the final report of mathematical progress written by Misha Kapovich, Arun Ram, and Monica Vazirani following the workshop.

- Find the MV polytope of an MV cycle from the Littelmann path data. A group centered around Joel Kamnitzer, James Parkinson, and Jacqui Ramagge formed around this problem, and they have a conjectured solution they are testing.

- More generally: understand the different combinatorial models involved (such as Knutson-Tao honeycombs, MV polytopes, Littelmann path models, canonical bases), provide a dictionary between them, and lay the groundwork to enable researchers to apply these tools toward a host of related problems. Substantial progress was made toward pieces of this during the workshop.

- Give an analogue of buildings for complex reflection groups. Arun Ram proposed a solution of what the building should be. The theory of p-compact groups says that there is a "p-compact group" X corresponding to each $\mathbb{Z}_p$-reflection group and that this p-compact group X contains a maximal torus T. The "quotient" X/T is an analogue of the flag variety corresponding to the reflection group. This is a starting point, but that does not mean we know what the building looks like or how to extract information from it. Many directions and open problems follow.

- Give an analogue of buildings for noncrystallographic reflection groups. In the case of a noncrystallographic reflection group, it is known how to construct the graded Hecke algebra. The graded Hecke algebra is a "degeneration" of the affine Hecke algebra that contains most of the information of the affine Hecke algebra. One can write a Satake map for the graded Hecke algebra and use this to compute numbers of triangles in the corresponding building, even though one does not know a construction of this building. There are inequalities that one can read off the Schubert calculus.

- Make an explicit correspondence between triangles and hives. Joel Kamnitzer explained a beautiful proposed solution to this problem using an action of three copies of the affine Grassmannian on three wedge powers, and a "fake moment map" on these triples.

- Understand to what extent the inclusions $C_{\mathrm{Hecke}} \subseteq C_{\mathrm{Rep}} \subseteq C_{\mathrm{Tri}}$ are strict, where C_{Hecke} is the cone of structure constants of the spherical Hecke algebra, C_{Rep} is the cone of tensor product multiplicities, and C_{Tri} is the cone of triangles in the building.

- Examine and compare the different approaches to the saturation theorem, with an emphasis on the role of buildings, to get more precise answers (in all types) and improve the proofs, and possibly also make a sensible Horn conjecture in other types.

- Find an analogue of hives outside the type A case.

- What is the analogue of stretching of paths on the level of points in the affine flag variety or the loop Grassmannian?

- Compute when general affine Deligne-Lusztig varieties in the affine flag variety are nonempty and, if possible, compute their dimension.

- In what way can the combinatorics of path models and MV cycles be applied to the Langlands program? David Nadler explained that, from the point of view of the geometric Langlands program, the geometric Satake correspondence ought to be lifted to an equivalence of categories.

- What is the relation between the moment map and sector retraction? Both of these operations take T-fixed points to weights and closed sets to convex hulls.

2 Questions and Conjectures from Misha Kapovich's overview lecture

More information on the following questions and conjectures can be found in the notes on Misha Kapovich's lecture, "Overview of connections between buildings and representation theory, and open problems."

Question. What are the restrictions on the side-length vectors of a triangle in a Euclidean building?

Question. Is there any irredundancy in the triangle inequalities?

Conjecture (Belkale-Kumar)**.** TI^{BK}—Belkale-Kumar inequalities—are irredundant.

Problem (Restricted triangle problem (Hecke problem))**.** What are the restrictions on the side-lengths $\lambda, \mu, \nu \in P_+^\vee$ so that there exists a triangle in the Euclidean building with special vertices and side-lengths λ, μ, ν?

Problem (Representation Theory problem Rep for $G^\vee$)**.** Find necessary and sufficient conditions on λ, μ, ν such that

$$(V(\lambda) \otimes V(\mu) \otimes V(\nu))^{G^\vee} \neq \{0\},$$

where $V(\gamma)$ is the irreducible representation of $G^\vee(\mathbb{C})$ with the highest weights γ.

Conjecture (Generalized saturation conjecture)**.** Saturation holds for all simply-laced root systems.

Conjecture (in the non-simply laced case (Knutson-Tao))**.** Suppose $(\lambda, \mu, \nu) \in C$ is so that $(\lambda + \mu + \nu)(t) = 1$ for all $t \in T$ such that $Z_{G(\mathbb{C})}(t)$ is semisimple. Here T is the maximal torus of $G(\mathbb{C})$ and $Z(t)$ denotes the centralizer of t. Then

$$(\lambda, \mu, \nu) \in C_{\text{Rep}}.$$

Conjecture (Kapovich, Millson)**.** For every root system, the intersection of C_{Rep} with the interior of the cone C is saturated. Moreover, if λ, μ, ν are regular, then

$$(\lambda, \mu, \nu) \in C_{\text{Rep}} \Longleftrightarrow (\lambda, \mu, \nu) \in C \cap L;$$

i.e.,

$$C_{\text{Rep}} \cap (\text{regular}) = C \cap (\text{regular}).$$

3 Problem proposed by Cristian Lenart

The category of crystals for complex semisimple Lie algebras is a *monoidal category* with an associative tensor product. The crystals $A \otimes B$ and $B \otimes A$ are isomorphic (via maps called *commutors*), but $(a, b) \mapsto (b, a)$ is not a commutor. Henriques and Kamnitzer [1] defined a commutor $\sigma_{A,B}$ based on an idea of Berenstein. Kamnitzer and Tingley [2] proved that the action of this commutor on the highest weight elements (which determines it) is given by *Kashiwara's involution* on the Verma crystal. In terms of *MV-polytopes*, the latter is given by the "negation" of a polytope. A different explicit description of the commutor, in terms of an abstract crystal, is given in [4].

The proposed problem below is related to a property of the mentioned commutor, and can also be viewed as a property of Kashiwara's involution. It involves the notion of the *key* of an element b in a crystal; this is denoted by $\kappa(b)$, and is a Weyl group element. The key is a generalization of the Lascoux-Schützenberger key [3], and it specifies the "smallest" *Demazure crystal* in which the element b lies (see, e.g., [7]). In terms of the alcove path model [5, 6], the key is a measure of the "folding" of the alcove path. As such, it is easily expressed in terms of the "foldings." One can also define the key recursively, in terms of an abstract crystal, just based on *Kashiwara's operators* F_i and E_i. Let b_λ be the highest weight element of the crystal B_λ, and b_λ^{low} its lowest weight element.

Proposition ([6])**.** The key $\kappa(b_\lambda)$ is the identity, and $\kappa(b_\lambda^{low})$ is the minimal representative of the coset $w_\circ W_\lambda$ (W_λ being the stabilizer of the weight λ). If $F_i(b)$ and $E_i(b)$ are both defined, then $\kappa(F_i(b)) = \kappa(b)$; otherwise, we have $\kappa(F_i(b)) = s_i\kappa(b)$, where s_i is the corresponding simple reflection.

Let the commutor σ_{B_λ,B_μ} be specified on highest weight elements by

$$b \otimes b_\mu \in B_\lambda \otimes B_\mu \mapsto c \otimes b_\lambda \in B_\mu \otimes B_\lambda \,.$$

Conjecture. We have $\kappa(c) = \kappa(b)^{-1}$ in the Weyl group W.

References

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