Polytopal models and tropical geometry

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Plan:

- Tensor product multiplicities
- Crystal bases

Let $G^\vee$ be a complex reductive group such as $\mathrm{GL}_3(\mathbb{C})$.
Then we have

$$\Delta_\lambda = \Delta_{1\times 1}^{\lambda_1-\lambda_2} \cdot \Delta_{2\times 2}^{\lambda_2-\lambda_3} \cdot \Delta_{3\times 3}^{\lambda_3} \in \mathbb{C}[G^\vee]$$

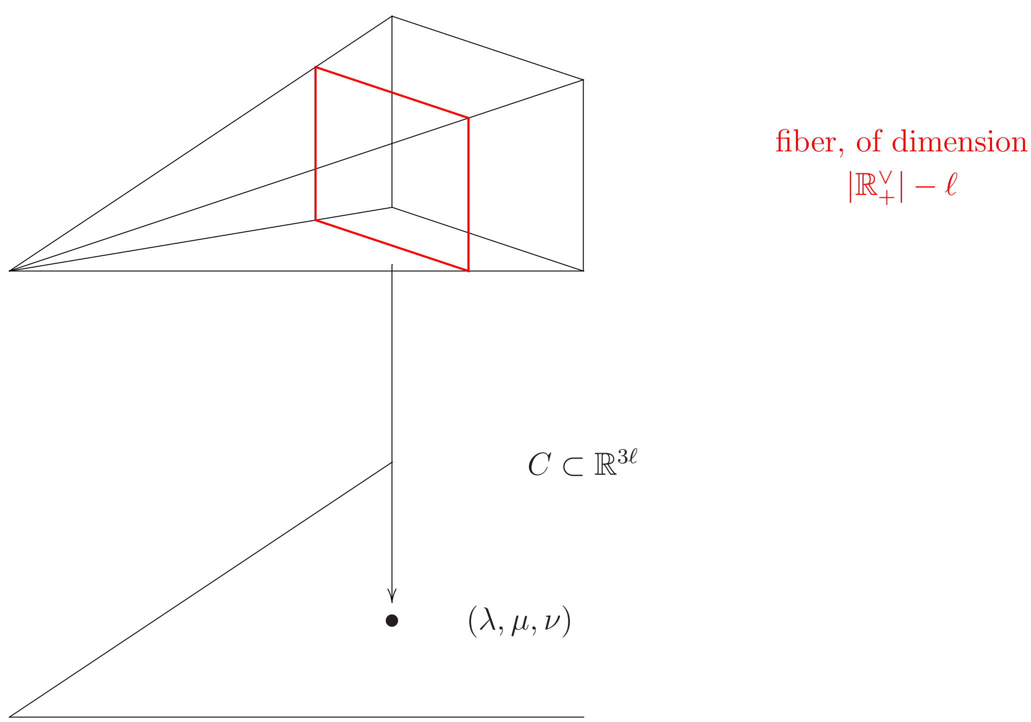
for the highest weight vector and $V_\lambda = \mathrm{Span}\{\Delta_\lambda \lhd G^\vee\}$.

Problem. Compute the tensor product multiplicity $c_{\lambda\mu\nu} = \dim(V_\lambda \otimes V_\mu \otimes V_\nu)^{G^\vee}$.

Theorem (A. Zelevinsky, A. Berenstein, ∼1999)**.** There is a convex polyhedral cone $C_{\underline{i}} \subset V = \mathbb{R}^{|R_+^\vee|+2\ell}$, where $\underline{i}$ is a reduced decomposition of the longest element in the Weyl group (for $\mathrm{GL}_3(\mathbb{C})$, $\underline{i} = (1\ 2\ 1)$ or $\underline{i} = (2\ 1\ 2)$), $R_+^\vee$ is the set of positive roots, and $\ell = \mathrm{rank}[G^\vee : G^\vee]$, and a linear map $\pi_{\underline{i}} : V \to (P^\vee \otimes \mathbb{R})^3$ such that

$$|\pi_{\underline{i}}^{-1}(\lambda,\mu,\nu) \cap (c_{\underline{i}})_{\mathbb{Z}}| = c_{\lambda\mu\nu}.$$

The associated picture is



Now let $G^\vee = \mathrm{GL}_3$, $\lambda = (\lambda_1,\lambda_2,\lambda_3)$ be non-decreasing, and $c_{\lambda\mu}^{\nu} = \mathrm{mult}(V_\mu, V_\lambda \otimes V_\nu) = c_{\lambda\mu^*\nu}$.

Theorem (I. M. Gelfand, A. Zelevinsky, ∼1985)**.** The number $c^{\nu}_{\lambda\mu}$ is the number of all triples $\left(\tilde{t}_1, \tilde{t}_2, \tilde{t}_3\right) \in (\mathbb{Z}_{\geq 0})^3$ such that

$$\lambda_1 - \lambda_2 \geq \tilde{t}_1, \qquad \lambda_2 - \lambda_3 \geq \tilde{t}_2 \qquad \lambda_2 - \lambda_3 \geq \tilde{t}_2 + \tilde{t}_3 - \tilde{t}_1,$$
$$\nu_1 - \nu_2 \geq \tilde{t}_2, \qquad \nu_1 - \nu_2 \geq \tilde{t}_2 + \tilde{t}_1 - \tilde{t}_3, \qquad \nu_2 - \nu_3 \geq \tilde{t}_3,$$

$$\lambda + \nu - \mu = \tilde{t}_1 \alpha_1^\vee + \tilde{t}_2(\alpha_1^\vee + \alpha_2^\vee) + \tilde{t}_3 \alpha_2^\vee = (\tilde{t}_1 + \tilde{t}_2)\alpha_1^\vee + (\tilde{t}_2 + \tilde{t}_3)\alpha_3^\vee, \tag{1}$$

whre the $\alpha_i^\vee$ are the co-roots.

Remarks.

1. For $\nu_1 - \nu_2$, $\nu_2 - \nu_3$ large, we obtain the Kostant partition function. In fact,
$$c^{\mu}_{\lambda\nu} \leq K_{\lambda,\mu-\nu} \leq \rho(\lambda + \nu - \mu).$$
2. In this case, we can produce a GT-pattern:
$$\begin{matrix} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & \lambda_2 + \tilde{t}_1 & & \lambda_3 + \tilde{t}_2 & \\ & & \lambda_3 + \tilde{t}_2 + \tilde{t}_3 & & \end{matrix}$$
Note that by the inequalities given in the theorem, we can "move down" the diagonals from left to right via $\geq$: $\lambda_1 \geq \lambda_2 + \tilde{t}_1 \geq \lambda_3 + \tilde{t}_2 + \tilde{t}_3$ and $\lambda_2 \geq \lambda_3 + \tilde{t}_2$. Similarly, since $(\tilde{t}_1, \tilde{t}_2, \tilde{t}_3) \in (\mathbb{Z}_{\geq 0})^3$, we can "move down" the diagonals from right to left via $\leq$: $\lambda_2 \leq \lambda_2 + \tilde{t}_1$ and $\lambda_3 \leq \lambda_3 + \tilde{t}_2 \leq \lambda_3 + \tilde{t}_2 + \tilde{t}_3$. (For B_n, we can take the longest word for B_{n-1}, and extend it uniquely to B_n to form analogous GT-patterns.)
3. If we set $\nu_1 - \nu_2 = \infty$ and $\nu_2 = \nu_3$, then we obtain the multiplicities for the Levi subgroup. This works in every Levi subgroup.

We now use tropical geometry to geometrize and generalize.
Rewrite $\nu - \mu$ in (1):
$$\lambda_2 - \lambda_3 + \min\left\{-\tilde{t}_2, \tilde{t}_1 - \tilde{t}_2 - \tilde{t}_3\right\}$$
Then in the (min, +)-algebra and writing $[t_i] = \tilde{t}_i$, $[a+b] = \min\left\{\tilde{a}, \tilde{b}\right\}$, and $[ab] = \tilde{a} + \tilde{b}$ (which implies $\left[\frac{1}{a}\right] = -\tilde{a}$), we can rewrite the inequalities from the theorem as follows ("outside the brackets is PL combinatorics; inside is ordinary algebraic geometry with no minus"):

$$\lambda_1 - \lambda_2 + \left[\frac{1}{t_1}\right] \geq 0, \qquad \lambda_2 - \lambda_3 + \left[\frac{t_1 + t_3}{t_2 t_3}\right] \geq 0,$$
$$\nu_1 - \nu_2 + \left[\frac{t_1 + t_3}{t_1 t_2}\right] \geq 0, \qquad \nu_2 - \nu_3 + \left[\frac{1}{t_3}\right] \geq 0$$
$$\lambda + \nu - \mu = [t_1 t_2]\alpha_1^\vee + [t_2 t_3]\alpha_2^\vee.$$

We can now interpret the expressions in brackets as functions on the upper triangular part of G.
For now,

$$G = \mathrm{GL}_3, \qquad U = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Let the coordinate chart on the unipotent group U be given by

$$x_1(t) = \begin{pmatrix} 1 & t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x_2(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

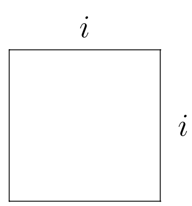
Then

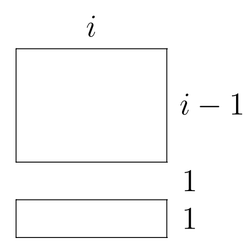
$$U = x_1(t_1)x_2(t_2)x_3(t_3) = \begin{pmatrix} 1 & t_1 + t_3 & t_1t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

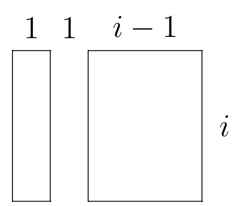
Write

$$\Delta_1 = t_1t_2, \qquad \Delta_2 = \det \begin{pmatrix} t_1 + t_3 & t_1t_2 \\ 1 & t_2 \end{pmatrix} = t_2t_3.$$

In general,

$\Delta_i = \det$ 

$\Delta'_i = \det$ 

${}'\Delta_i = \det$ 

Thus,

$$\Delta'_1 = t_2, \qquad \Delta'_2 = \det \begin{pmatrix} t_1 + t_3 & t_1t_2 \\ 0 & 1 \end{pmatrix} = t_1 + t_3,$$

$$\frac{\Delta'_1}{\Delta_1} = \left[\frac{1}{t_1}\right], \qquad \frac{\Delta'_2}{\Delta_2} = \left[\frac{t_1 + t_3}{t_2t_3}\right],$$

$$\frac{{}'\Delta_1}{\Delta_1} = \left[\frac{t_1 + t_3}{t_1t_2}\right], \qquad \frac{{}'\Delta_2}{\Delta_2} = \left[\frac{1}{t_3}\right],$$

$$\Delta_1 = [t_1t_2] \qquad \Delta_2 = [t_2t_3].$$

Generalizing, let G be an algebraic reductive group, B a Borel subgroup of G, and U the unipotent subgroup of G. Let $\{i\}$ be the vertices of the Dynkin diagram. Put $\mathbb{G}_m = \mathbb{C}^\times$, and consider

$$\begin{aligned} x_{\underline{i}} : (\mathbb{G}_m)^N &\to U_0 \\ (t_1, \dots, t_N) &\mapsto x_{i_1}(t_1) \cdots x_{i_N}(t_N), \end{aligned}$$

where $N = \dim U$.

Theorem (A. Zelevinsky, Fomin, A. Berenstein)**.** If $\underline{i}$ is a reduced word for w_0, then $x_{\underline{i}}$ is an open inclusion $(\mathbb{G}_m)^N \hookrightarrow U_0$, where $U_0 = U \cap B_- w_0 B_-$.

Aside: Cluster algebras are such charts that cover a target variety.

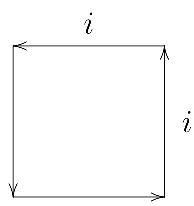
Define a principle minor $\Delta_\lambda \in \mathbb{C}[G]$ by

$$\Delta_\lambda(g) = \lambda(h),$$

where $g = U_- h U_+$, the character of the diagonal part.

Example. Let $G = \mathrm{GL}_n$. Then

$$\lambda = (\underbrace{1, \dots, 1}_{i}, 0, \dots, 0), \qquad \text{and} \qquad \Delta\omega_i = \det$$



Note that Δ_{ω_i} is the ordinary principal minor. For $w, \sigma \in W$ and ω_i a fundamental weight,

$$\Delta_{w\omega_i, \sigma\omega_i}(g) = \Delta_{\omega_i}\left(\overline{w}^{-1} g \overline{\sigma}\right).$$

If $w = s_{i_1} \cdots s_{i_m}$, then $\overline{w} = \overline{s_{i_1}} \cdots \overline{s_{i_m}}$, where

$$\overline{s_i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_i.$$

In this larger context,

$$\Delta_i = \Delta_{\omega_i, \omega_0\omega_i}, \qquad \Delta_i' = \Delta_{s_i\omega_i, \omega_0\omega_i},$$

$$'\Delta_i = \Delta_{\omega_i, \omega_0 s_i \omega_i},$$

and

$$C_{\underline{i}} = \left\{ \tilde{t} \in (\mathbb{Z}_{\geq 0})^{\dim U} : (\lambda, \alpha_i) + \left[\frac{\Delta_i'}{\Delta_i}\right] \geq 0, (\nu, \alpha_{i^*}) + \left[\frac{'\Delta_i}{\Delta_i}\right] \geq 0, \right.$$
$$\left. \lambda + \nu - \mu = \Sigma\, [\Delta_i] \alpha_i^\vee \right\},$$

where $\alpha_{i^*} = -\omega_0(\alpha_i)$.

Returning to the Levi discussion, we have

$$c^{\nu}_{\lambda\mu} = \dim V_\lambda(\mu - \nu, \nu),$$

where $V_\lambda(\mu - \nu, \nu) = \{v \in V_\lambda(\mu - \nu) : e^{(\nu, \alpha_i^\vee)+1}(v) = 0\}$.
Choose a subset I of the Dynkin diagram. We want ν such that

$$(\nu, \alpha_i^\vee) = \begin{cases} 0 & \text{if } i \notin I, \\ \infty & \text{if } i \in I, \end{cases}$$

where ∞ stands for a very large number.
Looking at the Levi $L = \langle x_i(\dots) \rangle$, we have

$$\mathrm{mult}(V_\beta : V_\lambda|_L) = \dim \mathrm{Hom}_L(V_\beta, V_\lambda).$$

This works even for Kac-Moody.

Key papers about this:

- tensor product, piecewise (Inventiones, 1999)
- Kazdan poasted years ago to arXiv geometric crystals II